IONIZATION AND EXCITATION IN THE UPPER LAYERS OF AN ATMOSPHERE.

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The Saha ionization formula has been deduced for the case of a gas in thermodynamical equilibrium. In a stellar atmosphere this condition is not fulfilled ; so it may be of interest to see what corrections must be applied to results deduced from the Saha formula. For the case of ionization by radiation (we may assume that ionization by collisions does not play an important part in the relevant atmospheric layers) Milne has deduced a formula containing probability factors for ionization through radiation of various frequencies. The factor by which the Saha result for the temperature $K = Px/(1 - x)$ has to be multiplied is *

$$\frac{\int \mathcal{J}\psi d\nu}{\int \psi e^{-h\nu/kT}\left(\frac{2h\nu^3}{c^2} + \mathcal{J}\right)d\nu},$$

where $4\pi\mathcal{J}$ is the total radiation of frequency ν falling from all sides on the atoms, ψ is the probability factor of ionization by ν radiation (*i.e.* the number of ionizations produced by $\mathcal{J}(\nu)d\nu$ is $\mathcal{J}(\nu)\psi(\nu)d\nu$ per atom), T is the gas temperature dominating the recombinations ; the integration extends over all ν from o to ∞. The first denominator term denotes the spontaneous recombinations, the second term denotes the stimulated ones, introduced by Milne. The probability factor $\psi(\nu)$ is connected with the absorption coefficient $k(\nu)$ by the relation $h\nu\psi = k$, because at every ionization a quantum $h\nu$ is absorbed.

In *B.A.N.*, 110,† this formula was applied to the case of black-body radiation at the effective temperature T_1 of the star, coming from below and ionizing the surface layer of the atmosphere, where the gas temperature is T_0. The result was, apart from a factor $\frac{1}{2}$ for the dilution of the radiation, the replacement of the temperature T_0 by T_1 in the exponential function in Saha's formula. In the same way in the Boltzmann factor for the number of excited atoms the effective temperature T_1 had to be used instead of the gas temperature T_0. This is in accordance with the result afterwards found by Russell ‡ that in order to represent the observations of line intensities for the Sun the logarithms of the computed fractions of atoms in different states of excitation must be multiplied by 0·85, whereas $T_0/T_1 = 0.83$.

These results could only be a second approximation, because now we know that the radiation passing through the upper layers of an atmosphere and emitted at the surface is not black-body radiation. By knowledge of the absorption coefficients we are able to compute the intensity distribution in the emitted spectrum. So with these improved values of $\mathcal{J}(\nu)$, which

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* Milne, *Phil. Mag.*, **47**, 209 ; Woltjer, *Physica*, V, 406 ; Pannekoek, *B.A.N.*, 110.
† *B.A.N.*, **3**, 207, 1926.　　　　　　‡ *Astroph. J.*, **70**, 53, 1929.

thus are based entirely on theoretical computation, we may compute the rate of ionization to a further approximation. We will first consider the high temperature stars, because for these the absorption coefficient is determined entirely by the hydrogen atoms and it is permissible to treat the case as being one of a pure hydrogen atmosphere.

Hydrogen Atmosphere.—The hydrogen absorption coefficient decreases with increasing wave-length between two consecutive band edges as ν^{-3}, but then at the next lower band edge jumps up to a higher value. So do the probability coefficients ψ, which are highest just below each band edge, decrease as ν^{-4} and jump at the next band edge to a high value again. The radiation, however, shows a dark absorption edge, hence has a small intensity precisely at the wave-lengths where k and ψ are high. So there is a certain compensation of the factors occurring in the integrals.

We consider only the surface layers where for $\mathcal{J}$ we may take the surface value of the radiation

$$\mathcal{J}(\nu)/E(\nu, 0) = \frac{2}{2 + \sqrt{3}} \left(1 + \frac{\sqrt{3}}{2} c \frac{\bar{k}}{k} \right),$$

where $4\pi E(\nu, 0)$ is the black-body radiation for the surface temperature and $c = \frac{1}{4} h\nu/kT (1 - e^{-h\nu/kT})^{-1}$. The absorption coefficients $k(\nu)$ and $\bar{k}$ occurring in this expression comprise the total absorption due to free-free transitions as well as to ionizations. To distinguish them we will denote the absorption coefficients due solely to ionization, to which are related the probability functions ψ, by k'. The factor $2/(2 + \sqrt{3})$ will be replaced by $1/2$. These absorption coefficients k and k' will be represented by

$$k = k_n(\nu_0/\nu)^3, \qquad k' = k_n'(\nu_0/\nu)^3,$$

where for ν_0 is taken the value for some standard wave-length $\lambda = 4000$ A., and k_n and k_n' are constant between two band edges, jumping from the highest value k_1 between ν_1 and ∞, k_2 between ν_2 and ν_1, down at every next band edge. For $E(\nu, 0)$ we take the Planck function for the surface temperature (which we write T instead of T_0) and put $h\nu/kT = y$.[*] We omit some coefficients which are the same in numerator and denominator, viz. $2h/c^2$ from the Planck function, and h from the value $\psi = k'/h\nu = (\nu_0/\nu)^3 k'_n/h\nu$. Then the correction factor required is

$$\frac{\int \frac{1}{2}\nu^3 (e^y - 1)^{-1} \left(1 + \frac{\sqrt{3}}{2} c \frac{\bar{k}}{k} \right) \frac{k'}{\nu} d\nu}{\int \nu^3 e^{-y} \frac{k'}{\nu} d\nu + \int \frac{1}{2}\nu^3 e^{-y} (e^y - 1)^{-1} \left(1 + \frac{\sqrt{3}}{2} c \frac{\bar{k}}{k} \right) \frac{k'}{\nu} d\nu}.$$

Each integral extends from 0 to ∞; because of the jumps in k and k' we divide it into an infinite number of integrals between the successive band edges. For the term with c, however, this is not necessary; because k and k' are varying in the same way, we may keep it as one integral between

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[*] The symbol k for the absorption coefficient has to be used also for the Boltzmann constant. Since in the latter case it always occurs in the combination kT no difficulties need to arise.

o and ∞, with (k'/k) as a factor in the integrand. For the numerator we may write now

$$N = \int \tfrac{1}{2}k_n'\nu_0{}^3(e^y - 1)^{-1}\frac{dv}{v} + \frac{\sqrt{3}}{4}\bar{k}\int c\left(\frac{k'}{k}\right)v^2(e^y - 1)^{-1}dv = N_1 + N_2,$$

$$N_1 = \tfrac{1}{2}k_1'\nu_0{}^3\int_{y_1}^{\infty}\frac{dy}{y(e^y - 1)} + \tfrac{1}{2}k_2'\nu_0{}^3\int_{y_2}^{y_1}\frac{dy}{y(e^y - 1)} + \ \ldots,$$

$$N_2 = \frac{\sqrt{3}}{4}\bar{k}\left(\frac{kT}{h}\right)^3\int_0^{\infty}c\left(\frac{k'}{k}\right)\frac{y^2 dy}{e^y - 1}.$$

In each of the terms of N_1 the constant k_n' is given by an infinite sum

$$k_n\nu_0{}^3 = \sum_n^{\infty}C_1 n^{-3}e^{-h\nu_1/n^2 kT},$$

where C_1 is composed of constants.* For each of the first terms this expression can be computed separately as the sum total of a convergent series. For the higher terms constituting an infinite tail, where y in each term changes little from y_{n-1} to y_n, the sum may be replaced by an integral (extending from $n - \tfrac{1}{2}$ to ∞, for which we may use as well the limits n to ∞)

$$k_n'\nu_0{}^3 = \int_n^{\infty}C_1 n^{-3}e^{-h\nu_1/n^2 kT}dn = C_1\frac{kT}{h\nu_1}(e^{y_n} - 1),$$

so that the successive higher terms of N_1 of the form

$$\tfrac{1}{2}k_n'\nu_0{}^3\int_{y_n}^{y_{n-1}}\frac{dy}{y(e^y - 1)} = \tfrac{1}{2}\frac{C_1}{y_1}\int_{y_n}^{y_{n-1}}\frac{dy}{y}$$

can be collected into one integral from $y = 0$ to $y = y_n$. So we have

$$N_1 = \tfrac{1}{2}\sum_1^n k_n'\nu_0{}^3\int_{y_n}^{y_{n-1}}\frac{dy}{y(e^y - 1)} + \tfrac{1}{2}\frac{C_1}{y_1}\int_0^{y_n}\frac{dy}{y}.$$

Thus N_1 is logarithmically infinite. This was to be expected, because it is well known that the Boltzmann distribution of hydrogen levels forms a non-convergent infinite series. In the partition function

$$\Sigma g_n \exp\left(-h\nu_1(1 - 1/n^2)/kT\right)$$

the exponential function with increasing n converges to a small constant fraction of the first term. The difficulty has been removed by Urey and by Fermi,† who took into account the circumstance that for atoms occupying a limited volume n cannot increase indefinitely, and the higher levels are gradually cut down by a volume function. Hence we are entitled to leave out the infinite number of highest levels, and we will do it by cutting off the tail at $n = \omega$, some big number, so that the limit o is replaced by y_ω. Then the last part of N_1 will be

$$\tfrac{1}{2}\frac{C_1}{y_1}(\ln y_n - \ln y_\omega) = \tfrac{1}{2}\frac{C_1}{y_1}\ln\frac{\omega^2}{n^2} = \frac{C_1}{y_1}\ln\frac{\omega}{n}.$$

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* *Publ. Amsterdam*, IV, pp. 12, 13.

† *Astroph. J.*, **59**, 1, 1924 ; *Z. f. Physik*, **26**, 54, 1924 ; cf. *Handb. d. Astrophysik*, III, **1**, 270.

Here y_1 corresponds to the lowest level of hydrogen, n is the number of the level where the separate terms cease and ω is some big number.

We have now to add N_2, the part of the numerator integral which is due to the increase of temperature in deeper layers. For the factor c we put its value $\frac{1}{4}y(1-e^{-y})^{-1}$. The ratio k'/k is nearly 1 for large ν, where the absorption through free-free transitions is small compared with ionization from the lowest levels. For small ν the free-free transitions play a dominant part. An approximate value for k'/k may be found if we assume for it the value it takes in the case of a smoothed absorption coefficient of a mixture with numerous band edges. In this case $k'/k = 1 - e^{-y}$; if we use this expression over the whole range of ν we get

$$N_2 = \frac{\sqrt{3}}{16}\bar{k}\left(\frac{kT}{h}\right)^3\int_0^\infty \frac{y^3 dy}{e^y-1} = \frac{\sqrt{3}}{16}\cdot\bar{k}\left(\frac{kT}{h}\right)^3\frac{\pi^4}{15}.$$

We now turn to the denominator integrals, the first of which represents the effect of the spontaneous captures. Just as in N_1 it is divided into a number of integrals between the successive band edges, whereas the sum total of all the tail terms from the nth term onward is collected into a single integral. The integrands differ from those in N_1 in that $(e^y-1)^{-1}$ is replaced by e^{-y}. So we have

$$D_1 = \sum_1^n k_n'\nu_0^3\int_{y_n}^{y_{n-1}}\frac{e^{-y}dy}{y} + \frac{C_1}{y_1}\int_0^{y_n}\frac{dy}{y}(1-e^{-y}).$$

For the second of these integrals there are no difficulties at the zero limit, because both the logarithm and the exponential integral tend to infinity and their difference tends to the Eulerian constant $\gamma = 0.5772$. So for it we write

$$\frac{C_1}{y_1}\{ln\,y_n + Ei(-y_n)+\gamma\}.$$

The other denominator integral D_2, produced by the stimulated recombinations, has the same form as the numerator N, with only the factor e^{-y} added. The effect of this factor is that for the first regions of ν, corresponding to the lowest levels, the integrals of D_2 are much smaller than those of N. For the higher levels, however, for $y \to 0$ the infinite tail of later terms approaches to the same value as in N. It likewise gives rise to a logarithmic infinity, which must be avoided by the introduction of the limit ω. This means that the transitions with low energy difference between very high levels and the ionized state depend, in either direction, chiefly on the low frequency radiation, whereas the one-sided spontaneous transitions are insignificant. We have

$$D_2 = \frac{1}{2}\sum_1^n k_n'\nu_0^3\int_{y_n}^{y_{n-1}}\frac{e^{-y}dy}{y(e^y-1)} + \frac{1}{2}\frac{C_1}{y_1}\int_0^{y_n}\frac{e^{-y}dy}{y} + \frac{\sqrt{3}}{16}\bar{k}\left(\frac{kT}{h}\right)^3\int_0^\infty\frac{y^3e^{-y}dy}{e^y-1}.$$

Each of the first integrals can be reduced

$$\int\frac{e^{-y}dy}{y(e^y-1)} = \int\frac{dy}{y(e^y-1)} - \int\frac{e^{-y}dy}{y},$$

so that $D_2 = N - \frac{1}{2}D_1$, or $D_1 + D_2 = N + \frac{1}{2}D_1$. This holds in the same way for the sum total of the tail terms. For the third part of D_2, produced by the temperature increase in the deeper layers, we have

$$\int_0^\infty \frac{y^3 e^{-y} dy}{e^y - 1} = \frac{\pi^4}{15} - 6 = 0.4940.$$

For the numerical computation of the separate terms in N_1 integrals are needed of the form $\int_y^\infty \frac{dy}{y(e^y - 1)}$. They have been computed by Dr. J. R. Airey who kindly provided me with a manuscript copy of them. We will indicate them by $A(y)$. In D_1 the terms can be found as differences of exponential integrals $-Ei(-y)$.

TABLE I

Correction Factor $\dfrac{N}{D}$ to the Saha Ionization Formula for Hydrogen Stars

$$\frac{x}{1-x}P = K\frac{N}{D}$$

	25200°		16800°		12600°		10080°	
	Num.	Den.	Num.	Den.	Num.	Den.	Num.	Den.
1st term	107	215	205	410	256	512	460	920
2nd ,,	58	108	100	196	151	298	210	418
3rd ,,	44	72	80	140	120	220	163	308
4th ,	34	47	62	94	94	154	130	223
5th ,,	27	34	49	67	76	110	105	160
6th ,,	22	26	40	51	62	84	87	122
Term with $\bar{k}$	15	1	32	2	85	6	169	13
Tail for N and D_2 $\begin{cases} \omega = 30 \end{cases}$	197		363		557		778	
60	282		518		796		1112	
120	367		673		1035		1446	
Tail of D_1	20		55		110		188	
$N : D$	504 : 720		931 : 1378		1401 : 2051		2102 : 3130	
,,	589 : 805		1086 : 1533		1640 : 2290		2436 : 3464	
,,	674 : 890		1241 : 1688		1879 : 2529		2770 : 3798	
N/D	0.70		0.68		0.68		0.67	
,,	0.73		0.71		0.72		0.70	
,,	0.76		0.74		0.74		0.73	

The computations have been made for temperatures 25200°, 16800°, 12600° and 10080° ; the first six integrals were computed separately, and the sum total was taken of the tail from the 6th level downward. For the large number ω where the tail was cut off, three values 30, 60, 120 were taken, hence $\omega/n = 5$, 10, 20. The limiting values $y_1 \ldots y_6$ of the integration domains are found by (2.3026×13.54) $(5040/T)$ divided by 1, 4, $\ldots$ 36. The results are given in Table I. The separate parts of numerator and denominator are given there in unit 0.00001. The correction factor for all these cases comes out about 0.70.

The difference between this result and the former result of *B.A.N.*, 110,

can easily be traced by means of the separate terms.   In the former treatment only ionizations from the lowest level were considered.   Here we see that for hydrogen the contributions of the higher levels decrease, but slowly, and if the number of levels actually occupied rises above 20 or 30 (which is certainly the case for giant and supergiant stars) the contribution of all these higher levels begins to exceed that of the lowest level.   For the lowest level $N_1/D_1 = \frac{1}{2}$;   for the highest levels, where both ionization and recombination are chiefly determined by the radiation $N_1/D_2 = 1$.   So with the increasing importance of the higher levels the ratio goes up from $\frac{1}{2}$ to 1.

The other difference is due to the increase of temperature in the deeper layers.   The term produced by this increase (the term with $\bar{k}$) is unimportant in the present computation.   The bright parts of the spectrum (at the long wave side of the band edges), where light from deep layers is emitted, have a small absorption coefficient and a small ψ, hence they contribute little to the ionization, whereas the parts that could contribute most are dark and their radiation comes only from the superficial layers. In the former computation the emitted light was supposed to be unweakened black-body radiation, all coming from rather deep layers.   If we introduce the former suppositions into the present formulæ, hence taking $\mathcal{J} = E_0(1 + c)$ and using only the lowest level, we have

$$\frac{N}{D_1 + D_2} = \frac{\frac{1}{2}\int \dfrac{dy}{y(e^y - 1)} + \frac{1}{8}\int \dfrac{e^y dy}{(e^y - 1)^2}}{\int \dfrac{e^{-y}dy}{y} + \frac{1}{2}\int \dfrac{e^{-y} dy}{y(e^y - 1)} + \frac{1}{8}\int \dfrac{dy}{(e^y - 1)^2}},$$

all integrals being taken from y_1 to ∞, where y_1 is always a large number $(31 \cdot 18 \times 5040/T)$.   The first integral in the numerator is Airey's integral, the first in the denominator the exponential integral, the second in the denominator is the difference of these two, which is imperceptible for high values of y_1.   From the two integrals representing the temperature increase with depth the second numerator term $= \frac{1}{8}(e^{y_1} - 1)^{-1}$, the third denominator integral $= \frac{1}{8}((e^{y_1} - 1)^{-1} - ln(1 - e^{-y_1})) = \frac{1}{8}(\frac{1}{2}e^{-2y_1} + \frac{3}{4}e^{-3y_1} + \ldots)$.   Numerical computation gives for $25200°$ ($y_1 = 6 \cdot 235$), and for $10080°$ ($y_1 = 15 \cdot 59$) :

$$\frac{N}{D} = \frac{0 \cdot 14 \times 10^{-3} + 0 \cdot 25 \times 10^{-3}}{0 \cdot 28 \times 10^{-3} + 0 \cdot 3 \times 10^{-6} + 0 \cdot 24 \times 10^{-6}} = 1 \cdot 4,$$

$$\frac{N}{D} = \frac{0 \cdot 52 \times 10^{-8} + 0 \cdot 21 \times 10^{-7}}{1 \cdot 04 \times 10^{-8} + (10^{-14})} = 2 \cdot 5.$$

Here the second numerator term is even larger than the first and shows the influence of the deep hotter layers, which in the old computation was expressed by the replacement of the surface temperature T_0 by the effective temperature T_1.

There can be no doubt that the new computation, showing a very small influence of the temperature increase only, is more correct.   We have to

consider, however, that our computation relates to the uppermost layers; in the deeper layers the radiation comes from all sides and the temperature is higher; so the ionization computed with the old formula may be nearly right for the mean of the relevant atmospheric layers.

The Solar Atmosphere.—The solar atmosphere is taken here as the type of a low-temperature atmosphere where the absorption is determined by a mixture of a large number of elements (metals). We assume that the number of levels and band edges is so large that for the dependence of the absorption coefficient on wave-length we may take a smoothed function. Taking the Gaunt factors for the free-free transition as well as for the ionizations 1, we have

$$k' = C_0 \left(\frac{\nu_0}{\nu}\right)^3 (e^{h\nu/kT} - 1) ; \qquad k = C_0 \left(\frac{\nu_0}{\nu}\right)^3 e^{h\nu/kT}.$$

We will consider the ionization of an element that has such a large number of levels that for the ionization probability ψ we may take the same smoothed form of k' as for the total mixture. Then we have

$$\psi = C_0 \frac{\nu_0^3}{h\nu^4}(e^{h\nu/kT} - 1) ; \qquad \mathcal{J} = \tfrac{1}{2}E_0\left(1 + \frac{\sqrt{3}}{2}c\frac{\bar{k}}{k}\right).$$

Omitting the same common factors as in the previous case we write for the correction factor

$$\frac{C_0\nu_0^3\left[\int\frac{1}{2}\nu^3(e^y - 1)^{-1}\frac{d\nu}{\nu^4}(e^y - 1) + \frac{\sqrt{3}}{16}\bar{k}\int\nu^3(e^y - 1)^{-2}\frac{d\nu}{\nu}y\left(\frac{k'}{k}\right)\right]}{C_0\nu_0^3\int\nu^3 e^{-y}\frac{d\nu}{\nu^4}(e^y - 1) + C_0\nu_0^3\int\frac{1}{2}\nu^3 e^{-y}(e^y - 1)^{-1}\frac{d\nu}{\nu^4}(e^y - 1) + \frac{\sqrt{3}}{16}\bar{k}\int\nu^3 e^{-y}(e^y - 1)^{-2}\frac{d\nu}{\nu}y\left(\frac{k'}{k}\right)}.$$

Considering that $\bar{k} = C_0\left(\frac{h\nu_0}{kT}\right)^3\frac{1}{0.873}$ [*] and putting $\frac{k'}{k} = 1 - e^{-y}$, we find

$$\frac{\int\frac{1}{2}\frac{dy}{y} + \frac{\sqrt{3}}{16}\frac{1}{0.873}\int\frac{y^3 e^{-y}dy}{e^y - 1}}{\int\frac{1 - e^{-y}}{y}dy + \int\frac{1}{2}\frac{e^{-y}dy}{y} + \frac{\sqrt{3}}{16}\frac{1}{0.873}\int\frac{y^3 e^{-2y}dy}{e^y - 1}}.$$

The limits of integration are 0 and ∞. At the zero limit we have the same difficulty as in the previous case, since the first numerator and the second denominator integral are infinite for this limit. Here also the infinity must be removed by cutting off the higher levels at some value of $y = \delta$ (corresponding to $h\nu_1/\omega^2 kT$ in the hydrogen case).

For the upper limit ∞ the first integrals in numerator and denominator both become infinite. This infinity, however, is not real, since it is introduced by our assuming a smoothed formula continually going up for higher ν, whereas the real k, above the last band edge belonging to the lowest level, decreases as ν^{-3}. So we may use the smoothed function as far as ν_1 corre-

[*] *Publ. Amst.*, IV, p. 32.

sponding to the lowest level, and then from $Y = h\nu_1/kT$ up to ∞ we have to use the value $k_1(\nu_0/\nu)^3$. For the latter domain of integration the integrals are

$$\tfrac{1}{2}k_1\nu_0{}^3\int_Y^\infty \frac{dy}{y(e^y-1)} = \tfrac{1}{2}k_1\nu_0{}^3 A(Y)\,; \qquad k_1\nu_0{}^3\int_Y^\infty \frac{e^{-y}dy}{y} = -k_1\nu_0{}^3 Ei(-Y)\,;$$

$$\tfrac{1}{2}k_1\nu_0{}^3\int_Y^\infty \frac{e^{-y}dy}{y(e^y-1)} = \tfrac{1}{2}k_1\nu_0{}^3\{A(Y)+Ei(-Y)\},$$

where, after division by the factors omitted above,

$$k_1\nu_0{}^3 = C_1/C_0\{g_1 e^{h\nu_1/kT} + g_2 e^{h\nu_2/kT} + \ \ldots\}.$$

The last integrals in numerator and denominator represent the influence of temperature increase. Since we have

$$\int_0^\infty \frac{y^3 e^{-y}dy}{e^y-1} = \tfrac{1}{15}\pi^4 - 6 = 0.4940 \quad \text{and} \quad \int_0^\infty \frac{y^3 e^{-2y}dy}{e^y-1} = \tfrac{1}{15}\pi^4 - 6\tfrac{3}{8} = 0.1190,$$

their numerical values are 0.061 and 0.015. So now the correction factor is

$$\frac{N}{D} = \frac{\tfrac{1}{2}k_1\nu_0{}^3 \cdot A(Y) + \tfrac{1}{2}(\ln Y - \ln\delta) + 0.061}{-k_1\nu_0{}^3 Ei(-Y) + \ln Y + Ei(-Y) - (\ln\delta + Ei(-\delta)) + \tfrac{1}{2}k_1\nu_0{}^3\{A(Y)+Ei(-Y)\}} \\ -\tfrac{1}{2}Ei(-Y) + \tfrac{1}{2}Ei(-\delta) + 0.015,$$

where for small values of δ, $\ln\delta + Ei(-\delta) = \delta - \gamma$.

For a numerical example we take the case of Ca for the solar temperature

$$5040/T = 0.87\,; \qquad C_1/C_0 = [1.55]\,; \qquad V = 6.09, \qquad Y = 0.87V \times 2.30 = 12.18,$$
$$k_1\nu_0{}^3 = [4.98], \quad A(Y) = [0.583 - 7].$$

For δ we take 0.12 and 0.06. Then we have

N		D_1		D_2	
k_1 term	0.018	k_1 term	0.036	k_1 term	0.000
$\tfrac{1}{2}\ln Y$	1.250	$\ln Y + Ei(-Y)$	2.500	$-\tfrac{1}{2}Ei(-Y)$	0.000
$-\tfrac{1}{2}\ln\delta \begin{cases} +1.060 \\ +1.407 \end{cases}$		$\gamma - \delta \begin{cases} +0.457 \\ +0.517 \end{cases}$		$\tfrac{1}{2}Ei(-\delta) \begin{cases} 0.832 \\ 1.179 \end{cases}$	

and $N/D = 2.389/3.840 = 0.62$ and $2.736/4.247 = 0.64$ in the two cases for δ. Here again we see that the result is near to 0.5 when the first terms dominate, but that with increasing importance of the tail, *i.e.* for smaller values of δ, the ratio increases towards 1. The influence of the temperature increase with depth is very small. So qualitatively the results are the same as in the case of hydrogen absorption. And here also we have to consider that the observed intensities of lines are not determined by the ionization of this uppermost layer only but by the ionization in all the deeper relevant layers too.

Excitation in the Upper Layers.—We take the simple case of a second level, which is fed by transitions from the first to the second level. The radiation ν_{12} producing these transitions is almost totally absorbed when reaching the upper layers; its intensity may be denoted by $I = rI(0)$ (r = residual fraction). If in the equilibrium condition, expressed in

Einstein probabilities $n_1 B_{12} I = n_2(A_{21} + B_{21} I)$, we introduce the values of these probabilities we find

$$\frac{n_2}{n_1} = \frac{p_2}{p_1}\frac{I}{2h\nu^3/c^2 + I} = \frac{p_2}{p_1}\frac{r}{e^{h\nu/kT} - 1 + r} \sim \frac{p_2}{p_1}re^{-h\nu/kT}.$$

Strictly speaking we should divide up the realm of the absorption line ν_{12} in its separate $d\nu$, and take the sum total of the results. Since, however, for small r $1/r \sim \sqrt{B_{12}}$ (because the diffusion coefficient $s \sim B_{12}$ and $r \sim \sqrt{k/(k+s)}$) the central parts with large s and B contribute most to the result, we may restrict ourselves to these central parts and make use of an average r for these parts. Moreover, the fraction of atoms occupying the 2nd level will deviate from this result because of the interaction of this level with the ionized state.

This holds for the uppermost layer. For other layers the intensity (in the case of two vertical radiation streams) is given by *

$$E_0\left(1 + ct - \frac{m - c/m}{1 + m}e^{-mt}\right), \qquad m = \sqrt{(k+s)/k}.$$

Since for the central parts of the line, which chiefly are responsible for lifting the atoms into the second state of energy, s is extremely large, it follows that m is also very large, the exponential term decreases rapidly with t and is only perceptible in the highest layers. For small values already of the optical depth (measured by the mean absorption) the ν_{12} radiation is given by $E_0(1 + ct)$, *i.e.* by black-body radiation of the local temperature. Everywhere in the atmosphere, except in an extreme thin top layer of an optical thickness of $\sqrt{k/s}$, the exciting radiation is black-body radiation, first of temperature T_0 and then increasing downward with increasing temperature. This is obvious, because through the strong absorption coefficient an atom is reached by ν_{12} radiation from the immediate neighbourhood only. Hence, except for that thin surface layer, the number of excited atoms is given for the upper layers by $p_2/p_1 e^{-h\nu_{12}/kT_0}$, and then for deeper layers by $p_2/p_1 \cdot e^{-h\nu_{12}/kT}$.

The observed intensity of an absorption line depends on the breadth of the profile where the residual intensity r is nearly $\frac{1}{2}$. These parts of the profile are determined by layers with an optical depth of nearly $\frac{1}{3}$, say between 0 and 1. In each layer the excitation depends on the local temperature in this layer. Since at an optical depth $\frac{2}{3}$ the local temperature is the effective temperature itself, we may say that the mean temperature determining the excitation and appearing in the exponent of the Boltzmann function, is somewhat lower than the effective temperature, but higher than the surface temperature. The old formula $\frac{1}{2}\exp(h\nu/kT_1)$, though not correct now as to the surface layers, will give results of the right order of magnitude for a line produced by the total atmosphere.

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