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Critical remarks on the central intensity in Fraunhofer lines, by *A. Pannekoek*.

1. In our investigation of the theoretical contours of Fraunhofer lines [1]) it was found that lines of moderate and great strength should be completely black in their centre, the central intensity falling below 1% of the background intensity. This is entirely in contradiction to the photometric measures of different authors. For the green *Mg* triplet H. H. PLASKETT [2]) finds central intensities 0·23, 0·22 and 0·19. For the K line of *Ca* MINNAERT [3]) finds 0·07, for $H\alpha$ 0·15. For a number of lines in the blue-violet of Rowland intensity > 10 WOOLLEY [4]) finds central intensities 0·10 — 0·20; for fainter lines it is usually above 0·20. We do not know of any line in the solar spectrum for which a lower central intensity has been measured.

Part of the discrepancy certainly must be attributed to instrumental errors. The scattering of light in the instrument caused by imperfections of the prisms and the gratings used, especially by ghosts in the latter case, as well as the broadening of a narrow line by diffraction, by finite slit width and by diffusion in the sensitive film increase the central intensity of a line so that even in a real black centre of a line a finite intensity is produced. The amount of this stray light may be determined if lines are measured for which we are sure that the centre must be completely black. In this way S. A. KORFF [5]), by means of artificial absorption lines of sodium, found an intensity 0·12 instead of zero. The measures of the atmospheric *A* band by H. VON KLÜBER [6]) (dispersion 1·37 *A* per mm) show a central intensity which for the strongest lines decreases to 0·10, but not lower; here too a theoretical intensity zero could be expected. In a careful research on the true contours of some Fraunhofer lines by means of an interferometer placed before the spectrograph (dispersion 0·86 *A* per mm) C. D. SHANE [1]) determined the scattered light, ghosts etc. amounting to 0·09. Hence we cannot be sure of the reality of central intensities measured below 0·1; and it is quite well possible that in the case of strong lines the real central intensity may be of the order of only 0·01, though an exact determination will be a matter of much practical difficulty. On the other hand it cannot be doubted that for a number of rather strong lines the central intensity is far above this value, and may reach 0·2 and more. In the strongest line measured by SHANE (8 on the scale of Rowland) the corrected central intensity was 0·17; here the wings are strong already and after our theoretical results the central intensity could not be more than 0·01.

2. This theory rests on the assumption of two processes in the solar atmosphere, usually called diffusion and absorption, producing the Fraunhofer line. The ratio of the absorption and the diffusion coefficients determines the residual intensity of the light emitted at the surface. The absorption is due to collisions of electrons with the atoms. The general absorption extending over the continuous spectrum is partly caused by a direct interchange of radiation energy with translation energy of the colliding electron, partly by ionization and recombination. In the centre of a Fraunhofer line it is increased by the interchange of quantum energy of the excited atom with translation energy of the colliding electrons (inelastic and hyperelastic collisions). These absorptions depend on the number of collisions; in the upper layers of the atmosphere with

1) *Monthly Notices R A S* **91**. 139 (1930).
2) *M. N. R A S.* **91**. 870 (1931).
3) *Zschr. f. Physik* **45**. 610 (1927).
4) *Annals Solar Physics Obs.* Cambridge III. part 2 (1933.)
5) *Aph. J.* **76**. 135 (*Contr. Mt Wilson* 455) (1932).
6) *Zschr. f. Astroph.* **6**. 173 (1932).

1) *Lick Bulletin* 449 (1932).

decreasing electron density the number of collisions and the coefficient of absorption decrease and vanish at the surface. For a wave length in the centre of a line, in consequence of the high diffusion coefficient, all the light emitted at the surface comes from the uppermost layers, where the absorption coefficient by the lack of collisions is extremely small. This is the reason why theory predicts a residual intensity of nearly zero for this wave length.

In a paper "*The Effect of Collisions on the Central Intensity of the Fraunhofer Lines*" [1]) R. v. d. Riet Woolley, contrary to our results, finds "that the collisions make an appreciable contribution to the central intensity of the line, notwithstanding the fact that there are practically no collisions in the uppermost layers of the atmosphere". The difference between our results is due to the fact that Woolley introduces a density dependence only for the second source of general absorption, the transformation of quantum energy of the atom by inelastic and hyper-elastic collisions (the term εl in Eddington's notation), and assumes the other source, the general absorption, (k in Eddington's notation) to be constant over all the layers. Then for the topmost layers, which alone are active for the line centre, the limiting value of the absorption coefficient is not zero, but k; hence a finite result for the central intensity, depending on Eddington's $\eta = l/k$ must be found.

Now there can be no doubt that in the solar atmosphere the general absorption coefficient decreases to zero in the highest layers. Its first part, due to free-free transitions, contains the electron pressure simply as a factor. Its second and chief part, due to ionization processes, does not contain this electron pressure directly. It depends, however, on the number of non-ionized but easily ionizable atoms, which has been shown in the paper already quoted (Appendix, p. 165) to contain the electron pressure as a factor also. Hence Woolley's equation cannot represent the real conditions. The use of a constant general absorption coefficient in many cases may have practical advantages, where by an easy and short formula it gives a result of sufficient approximation. It cannot be used, however, to disprove the result of an exact computation based upon the real variation of the absorption coefficient with electron pressure.

3. The central intensity is not the sole discrepancy between theory and observation. The equations found in our paper on the theoretical contours are derived only for the case of vertical energy streams, i.e. for lines observed on the centre of the solar disc. For points near the limb we have to introduce into the equations for $dy/d\xi$ and $dz/d\xi$ on p. 143 a coefficient $\sec\vartheta$ into the denominator:

$$\frac{dy}{d\xi\sec\vartheta} = (k_0+s_0)\,z\,; \quad \frac{dz}{d\xi\sec\vartheta} = \left(k_0 + s_0\frac{x}{x+q}\right)(y-2E)$$

Instead of substituting, as was done for the vertical rays,

$$dx = s_0\,d\xi\,; \quad \frac{k_0}{s_0} = \frac{\varkappa_0}{s_0^{\,2}}\frac{g}{p'}\,x = \alpha x\,; \quad \frac{q}{x} = \frac{Q}{P}\,; \quad 2E = 2E_0\left(1 + \frac{1}{2}c\,\frac{\bar{\varkappa}}{\varkappa_0}\,\alpha x^2\right)$$

we introduce now $x\sec\vartheta = x'$; $q\sec\vartheta = q'$; $\alpha\cos\vartheta = \alpha'$ and the equations which replace equation (6) in the former paper, are

$$\frac{dy}{dx'} = (1+\alpha'x')\,z\,; \quad \frac{dz}{dx'} = \left(\frac{x'}{x'+q'} + \alpha'x'\right)\left(y - 2E_0\left(1 + \frac{1}{2}c\,\frac{\bar{\varkappa}}{\varkappa_0}\,\alpha'x'^2\cos\vartheta\right)\right).$$

They have the same form as (6) and the tables of solutions, given in the former paper, hold also for this case, provided that we take the dashed values as arguments. Since $\alpha'q'^2 = \alpha q^2\sec\vartheta$, this means that in the case of $\cos\vartheta = 0{\cdot}1$) we look in Table II (p. 147) for αq^2 10 times larger, at the argument q 10 times larger or α 10 times smaller. Furthermore the coefficient of the temperature term found there should be taken 10 times smaller, in the same way as in the continuous background intensity the temperature term is 10 times smaller. It appears that the residual intensity r at the sun's border is found smaller than it is in the centre, i.e. the line should appear blacker and broader at the border. This is just the reverse of what is observed.

The reason why theory should give this result, is easily to see. The light ray emitted at the surface comes from a certain range of optical depths, determined by its diffusion coefficient. If it passes the atmosphere obliquely the same range of optical depths measured along the ray is situated in higher layers with a smaller electron pressure, where in consequence the continuous absorption coefficient is smaller. Hence the intensity in the oblique ray is smaller and it appears blacker.

That such a greater blackness is not observed indicates that the mechanism, which throws light into the central parts of the Fraunhofer lines, where

[1]) *Zschr. f. Aph.* V. 67 (1932).

by pure diffusion they would appear entirely black, cannot decrease to zero in the uppermost layers of the atmosphere. It cannot be the absorption due to electron collisions, though this absorption is certainly present and plays an important part in the deeper layers. Superimposed upon it there must work another mechanism, which does not decrease in the highest layers of the atmosphere and which also must be the cause of the observed finite central intensities in the Fraunhofer lines.

4. In a paper "*Zur Deutung der Intensitätsverteilung in den Fraunhoferschen Linien*" [1]) A. UNSÖLD suggests that such a mechanism may be found in the interactions with higher atomic levels. If a Fraunhofer line is produced by the transitions between the levels 1 and 2, then part of the atoms in state 2 does not directly return to state 1, but is lifted to state 3. Returning then to state 2 the atom will be capable to emit the radiation 1—2 and to produce light in the centre of the Fraunhofer line, without an immediately preceding absorption process. UNSÖLD says (treating the case of the D lines of Na): "Die einem solchen Prozess entsprechende Reemission in den D-linien wird von dem ersten Absorptionsvorgang unabhängig sein und kann – wenigstens näherungsweise – einfach durch Anwendung des Kirchhoff'schen Satzes abgeschätzt werden" (p. 320). The frequency of occurrence of these transitions 2–3 determines the "absorption coefficient" (his $\varkappa$), the frequency of the transitions 1—2 determines the "diffusion coefficient" (his σ). The residual intensity is of the order of magnitude $\sqrt{k/\sigma}$; UNSÖLD makes an estimate of the relative number of these transitions and finds $k/\sigma = 0{\cdot}047$; hence the residual intensity comes out 0·22.

There may be some doubt whether we are allowed to consider the transitions 2—3 as producing an "absorption" mechanism ruled by KIRCHHOFF's law. But it is not necessary to argue about this question, because the problem can be treated in a more rigorous way.

We follow UNSÖLD in assuming two kinds of transitions 1—2—1 and 2—3—2, each emitting and absorbing its own frequency, ν_{12} and ν_{23}; the Einstein probability coefficients A and B are provided with the same indices. We consider only vertical radiations, upward (I) and downward (I'), and we introduce a homogeneous depth by $dx = -\rho dh$. The number of atoms per unit mass in state 1, 2, 3, is n_1, n_2, n_3. Then the equations of radiation are:

$$
\begin{aligned}
dI_{12}/dx &= (n_1 B_{12} I_{12} - \tfrac{1}{2} n_2 A_{21})\, h\nu_{12} \\
dI'_{12}/dx &= -\ (n_1 B_{12} I'_{12} - \tfrac{1}{2} n_2 A_{21})\, h\nu_{12} \\
dI_{23}/dx &= (n_2 B_{23} I_{23} - \tfrac{1}{2} n_3 A_{32})\, h\nu_{23} \\
dI'_{23}/dx &= -\ (n_2 B_{23} I'_{23} - \tfrac{1}{2} n_3 A_{32})\, h\nu_{23}
\end{aligned}
$$

and the equations of equilibrium, i.e. of the constancy of the numbers in each state are

for 1: $n_1 B_{12} (I_{12} + I'_{12}) = n_2 A_{21}$
for 2: $n_2 A_{21} + n_2 B_{23} (I_{23} + I'_{23}) = n_1 B_{12} (I_{12} +$
$+ I'_{12}) + n_3 A_{23}$
for 3: $n_3 A_{23} = n_2 B_{23} (I_{23} + I'_{23})$.

The second equation is simply found by adding the other two, so they reduce to the equations for state 1 and state 3, the former completing the set of equations for I_{12} and I'_{12}, the other those for I_{23} and I'_{23}. It appears that the two sets of equations are entirely independent, just as if the other transitions do not exist. This means that in the case assumed by UNSÖLD the transitions to and from a third state cannot change the results for the intensities of the radiation ν_{12} at all. In the ordinary theory the origin of the black emission produced by collisions is the continual rearrangement of the translation energies in the electron gas into a black body distribution. Such a rearrangement is lacking in the case of energy transferred to the third atomic state; what is given to it comes back unchanged and so there is no effect.

5. The case is different, however, if we could introduce the transitions 1—3 also. Then the three kinds of radiation would not be independent any more. If in an atom transitions are possible between the levels 1 and 2 and also between levels 2 and 3, then, it is true, direct transitions between levels 1 and 3 are forbidden. The atom may, however, proceed from state 1 to state 3 along another way, over a fourth level. Then cyclical transitions are possible in one or in the opposite direction. In the case of a sodium atom, where the transitions 1 S — 2 P produce the yellow doublet, the electron can return to its lowest state indirectly by the way 2 P — 3 D — 3 P — 1 S, or by 2 P — 2 S — 3 P — 1 S, and it can in the opposite direction, without absorption of yellow light, through 1 S — 3 P — 3 D — 2 P reach the level where it can emit the yellow doublet.

For each of the transitions 1—2, 2—3, 3—4, 1-... we have a system of equations such as:

$$
\begin{aligned}
dI_{12}/dx &= (n_1 B_{12} I_{12} - \tfrac{1}{2} n_2 A_{21})\, h\nu_{12} \\
dI'_{12}/dx &= (-\ n_1 B_{12} I'_{12} + \tfrac{1}{2} n_2 A_{21})\, h\nu_{12}
\end{aligned}
$$

or

$$
\begin{aligned}
dJ_{12}/dx &= n_1 B_{12} F_{12}\, h\nu_{12} \\
dF_{12}/dx &= (n_1 B_{12} J_{12} - n_2 A_{21})\, h\nu_{12}.
\end{aligned}
$$

[1]) *Zschr. f. Astrophysik* 4. 319 (1932).

where $J = I + I'$ is the total radiation intensity and $F = I - I'$ is the net stream.

The statistics for the 4 states afford the equations

$$\begin{aligned}
(n_2 A_{21} - n_1 B_{12} J_{12}) + (n_4 A_{41} - n_1 B_{14} J_{14}) &= 0 \\
(n_3 A_{32} - n_2 B_{23} J_{23}) - (n_2 A_{21} - n_1 B_{12} J_{12}) &= 0 \\
(n_4 A_{43} - n_3 B_{34} J_{34}) - (n_3 A_{32} - n_2 B_{23} J_{23}) &= 0 \\
(n_4 A_{41} - n_1 B_{14} J_{14}) + (n_4 A_{43} - n_3 B_{34} J_{34}) &= 0
\end{aligned}$$

which reduce to the equality

$$\begin{aligned} n_2 A_{21} - n_1 B_{12} J_{12} &= n_3 A_{32} - n_2 B_{23} J_{23} = \\ &= n_4 A_{43} - n_3 B_{34} J_{34} = n_1 B_{14} J_{14} - n_4 A_{41} = G/h. \end{aligned}$$

This quantity represents the excess of the number of cyclical transitions 1—4—3—2—1 over the number of the opposite cycle 1—2—3—4—1.

Introducing the macroscopic diffusion coefficients $n_1 B_{12}\, h\nu_{12} = s_{12}$; $n_2 B_{23}\, h\nu_{23} = s_{23}$, etc., and making use of the relations $A_{21} = \sigma_{12} B_{12}\, g_1/g_2$ where $\sigma_{12} = 2\, h\nu^3{}_{12}/c^2$, the cycle equation becomes

$$\left(\frac{n_2 g_1}{n_1 g_2}\sigma_{12} - J_{12}\right) s_{12}/\nu_{12} = \left(\frac{n_3 g_2}{n_2 g_3}\sigma_{23} - J_{23}\right) s_{23}/\nu_{23} = \left(\frac{n_4 g_3}{n_3 g_4}\sigma_{34} - J_{34}\right) s_{34}/\nu_{34} = -\left(\frac{n_4 g_1}{n_1 g_4}\sigma_{14} - J_{14}\right) s_{14}/\nu_{14} = G. \quad (2)$$

The equations for the radiation streams, where, for the sake of comparison, we will assume a real black body absorption and emission too, take the form (for each of the 4 transitions)

$$dJ_{12}/dx = (s_{12} + k)\, F_{12}$$

$$dF_{12}/dx = (s_{12} + k)\, J_{12} - \frac{n_2 g_1}{n_1 g_2} s_{12}\sigma_{12} - 2\, k\, E_{12}$$

Substituting here the expressions for equilibrium from equ (2) we find

$$\begin{aligned}
dJ_{12}/dx &= (s_{12} + k)\, F_{12} \quad ; \quad & dF_{12}/dx &= -\, G\nu_{12} + k\, (J_{12} - 2\, E_{12}) \\
dJ_{23}/dx &= (s_{23} + k)\, F_{23} \quad ; \quad & dF_{23}/dx &= -\, G\nu_{23} + k\, (J_{23} - 2\, E_{23}) \\
dJ_{34}/dx &= (s_{34} + k)\, F_{34} \quad ; \quad & dF_{34}/dx &= -\, G\nu_{34} + k\, (J_{34} - 2\, E_{34}) \\
dJ_{14}/dx &= (s_{14} + k)\, F_{14} \quad ; \quad & dF_{14}/dx &= +\, G\nu_{14} + k\, (J_{14} - 2\, E_{14})
\end{aligned} \quad (3)$$

In the usual equation without G it is the term with k in dF/dx which determines the residual intensity as a quantity of the order $\sqrt{k/(s+k)}$. Here it appears that in the absence of a real absorption k the terms with G act in the same way.

The value of G may be found by eliminating the unknown relative numbers of atoms in each state: n_2/n_1, n_3/n_2, n_4/n_3 from the equations (2). The resulting equation for G is of the third degree:

$$\frac{\nu_{12}}{s_{12}\,\sigma_{12}} \cdot \frac{\nu_{23}}{s_{13}\,\sigma_{13}} \cdot \frac{\nu_{34}}{s_{34}\sigma_{34}}\, G^3 + \left(\frac{\nu_{12}}{s_{12}\sigma_{12}}\frac{\nu_{23}}{s_{23}\sigma_{23}}\frac{J_{34}}{\sigma_{34}} + \ldots\right) G^2 + \left(\frac{\nu_{12}}{s_{12}\sigma_{12}}\frac{J_{23}J_{34}}{\sigma_{23}\sigma_{34}} + \ldots\right) G + \frac{\nu_{14}}{s_{14}\sigma_{14}}\, G = \frac{J_{14}}{\sigma_{14}} - \frac{J_{12}J_{23}J_{34}}{\sigma_{12}\,\sigma_{23}\,\sigma_{34}}. \quad (4)$$

This looks very complicated. The possibility of a solution appears by the following consideration. The J are intensities of radiation streams, at most equal to $2\,E$, the black body radiations of the same frequency; hence such ratios as J_{12}/σ_{12} are small fractions, of the order of magnitude of n_2/n_1, or exp. $(-\, h\nu_{12}/kT)$, and the right hand side of the equation is a small quantity. Furthermore the last term at the left hand side may be expected to be larger than the other terms of the first degree (the more so if s_{14} is smaller than the other s); the quantities $(\nu/s\sigma)\, G$ will be small fractions, so that the terms of higher degree may be neglected. In an estimate of the order of magnitude of G we may omit these lesser terms and write

$$\frac{\nu_{14}}{s_{14}\,\sigma_{14}}\, G = \frac{J_{14}}{\sigma_{14}} - \frac{J_{12}J_{23}J_{34}}{\sigma_{12}\,\sigma_{23}\,\sigma_{34}}$$

We denote $J/2E$ by r, the residual intensity, here taken as a function of the depth. Writing $2E = \sigma e^{-h\nu/kT}$, and considering that $\nu_{12}+\nu_{23}+\nu_{34} = \nu_{14}$ we find for $G\nu_{12}$, which chiefly interests us, because it determines F_{12}:

$$G\nu_{12} = \frac{\nu_{12}}{\nu_{14}}\, s_{14}\, (r_{14} - r_{12}\, r_{23}\, r_{34})\, 2\, E_{14}\,.$$

In the cases considered here ν_{12} will be a strong absorption line, whereas ν_{23}, ν_{34} and ν_{14} usually are much fainter lines. In the atmospheric layers r_{12} rapidly decreases to a small boundary value; r_{23}, r_{34} and r_{14} on the contrary will slowly decrease to rather high boundary values. If then the expression $r_{14} - r_{12}\, r_{23}\, r_{34}$, as to the order of magnitude, is replaced by $1 - r_{12}$, we obtain the equation

$$dF_{12}/dx = -\, G\nu_{12} = -\, \frac{\nu_{12}}{\nu_{14}}\frac{E_{14}}{E_{12}}\, s_{14}\, (2\,E_{12} - J_{12}),$$

entirely analogous to the absorption term, with $s_{14}\, \nu_{12}\, E_{14}/\nu_{14}\, E_{12}$ instead of the coefficient k. The resulting residual intensity, otherwise given by $\sqrt{k/s}$, comes out now

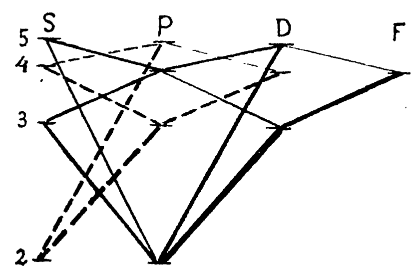
$$r_{12} = \sqrt{\frac{\nu_{12}}{\nu_{14}}\frac{E_{14}}{E_{12}}\frac{s_{14}}{s_{12}}}.$$

Unsöld supposed a mechanism, where part of the energy from level 2 P is lifted to higher levels, and then returns to 2 P. Such a mechanism, we saw, cannot change anything in the intensity of the doublet 1 S — 2 P. His suggestion may, however, afford a better result if applied to cyclical processes. In the deep layers, where the deviations from thermodynamic equilibrium are small the two opposite cycles of transitions are nearly equal. In the highest layers the intensity of the ν_{12} radiation (the doublet 1 S — 2 P) strongly decreases, the transitions 1 — 2, from 1 S to 2 P become scarcer, and the cycle 1—4—3—2—1 becomes dominant over the opposite cycle. Additional energy in this way is brought to the level 2 P, which increases the central brightness of the line ν_{12}. A more exact computation is necessary to determine how much it is increased.

In the value found above only the coefficient s_{14} occurs, because s_{14} was supposed to be the smallest coefficient, or rather, the first degree term with $1/s_{14}$ was supposed to be the largest. If one of the other coefficients e.g. s_{34} should be far the smallest then the first degree term with $1/s_{34}$ would be the largest and chief term and in the result the ratio of this coefficient and s_{12} would appear. It is clear that in the cycle of four transitions that with the smallest transition probability offers the greatest resistance and determines how much the intensity of this cyclical stream of energy surpasses the opposite cycle which is dimmed by the strong absorption in the line considered.

6. The transition probabilities which are necessary for this computation are for the higher levels known only in the case of hydrogen. We will treat here the case of $H\alpha$, which consists of three components, for each of which the possible transition cycles must be computed separately. We distinguish two kinds of cycles, the first kind of the form 1—2—3—4—1, the second kind of the form 1—2—4—3—1, 4 being in each case the highest energy level. The 7 possible cycles of the first kind with their transition probabilities, taken from F. G. Slack are indicated in the following table (the factor 10^7 is omitted).

2 S	2·26	3 P	0·180	4 S	0·074	5 P	0·497	2 S
2 S	2·26	3 P	0·720	4 D	0·0002	5 P	0·497	2 S
2 P	0·632	3 S	0·309	4 P	0·125	5 S	0·127	2 P
2 P	0·632	3 S	0·309	4 P	0·150	5 D	0·946	2 P
2 P	6·47	3 D	0·035	4 P	0·150	5 D	0·946	2 P
2 P	6·47	3 D	1·38	4 F	0·005	5 D	0·946	2 P
2 P	6·47	3 D	0·035	4 P	0·125	5 S	0·127	2 P



Of course analogous cycles are possible where still higher levels 6 S, 6 P, 6 D etc. come into play; because here the transition probabilities are smaller, it seems sufficient to consider only the cycles with the 5th quantum state as its highest level. Putting

$$\frac{s_{12}\sigma_{12}}{\nu_{12}} = \alpha_{12} = n_1 h A_{21}\, g_2/g_1\ ;\ \frac{s_{23}\sigma_{13}}{\nu_{13}} = \alpha_{13} = n_2 h A_{32}\, g_3/g_2\,,\ \text{etc.}\ldots.. J/\sigma = j = re^{-h\nu/kT}\,,$$

we can write the equation for G in a somewhat shorter way

$$G^3 + G^2\,(\alpha_{12} j_{12} + \alpha_{23} j_{23} + \alpha_{34} j_{34}) + G\,(\alpha_{12}\,\alpha_{23} j_{12} j_{23} + \alpha_{12}\,\alpha_{34} j_{12} j_{34} + \alpha_{23}\,\alpha_{34} j_{23} j_{34} + \alpha_{12}\,\alpha_{23}\,\alpha_{34}/\alpha_{14}) = \\ = \alpha_{12}\,\alpha_{23}\,\alpha_{34}\,(j_{14} - j_{12} j_{23} j_{34}).$$

Inserting the numerical values, and omitting the constant factors $h \times n_1$ (for the lowest level) $\times\ 10^7$, which disappear in the final result, we have for the 5th of the above cycles 2 P — 3 D — 4 P — 5 D —2P, taken as specimen:

$$\alpha_{12} = [1{\cdot}03]\ ^{1});\quad \alpha_{23} = [\bar{6}{\cdot}63];\quad \alpha_{34} = [\bar{7}{\cdot}27];\quad \alpha_{14} = [0{\cdot}20];$$

$$j_{12} = [-1{\cdot}69]\, r_{12};\quad j_{23} = [-0{\cdot}44)\, r_{23};\quad j_{34} = [-0{\cdot}25]\, r_{34};\quad j_{14} = [-2{\cdot}38]\, r_{14}.$$

$$G^3 + G^2 \{ [\bar{9}{\cdot}34]\, r_{12} + [\bar{6}{\cdot}19]\, r_{23} + [\bar{7}{\cdot}02]\, r_{34} \} \\ + G \{ [\bar{5}{\cdot}53]\, r_{12} r_{23} + [\bar{6}{\cdot}36]\, r_{12} r_{34} + [\bar{3}{\cdot}21]\, r_{23} r_{34} + [\bar{4}{\cdot}73] \} = [\bar{2}{\cdot}55]\,(r_{14} - r_{12} r_{23} r_{34}).$$

This equation shows a different character from that supposed in the general treatment of § 5 in that in the coefficient of G not the last but another term is largest. The transition with lowest probability is not

1) Numbers in brackets denote logarithms, $\bar{7}$ denotes 7—10.

1—4 but some other one, in this case 2—3, generally a transition where the azimuthal quantum number decreases (with increasing total quantum number). The equation is now more complicated; still the numerical values show that the higher powers of G may be neglected. If, moreover, we assume that the three other lines are weak, i.e. r_{23}, r_{34} and r_{14} up to the boundary remain large, and may be put 1, the equation takes the form

$$([\bar{6}\cdot 42]\, r_{12} + [\bar{4}\cdot 74])\, G = [\bar{2}\cdot 55]\, (1 - r_{12})$$

and $G\nu_{12}$, occurring in the equation for dF_{12}/dx has the value

$$G\nu_{12} = \frac{G s_{12}\sigma_{12}}{\alpha_{12}} = \frac{G s_{12}}{\alpha_{12}}\, 2 E_{12} e^{h\nu_{12}/kT} = \frac{[\bar{2}\cdot 55]\,(1-r_{12})}{[\bar{6}\cdot 42] r_{12} + [\bar{4}\cdot 74]} \cdot \frac{[\bar{1}\cdot 69]}{[\bar{1}\cdot 03]}\, 2 s_{12} E_{12} = [\bar{6}\cdot 79]\, \frac{(1-r_{12})\, 2 E_{12}}{r_{12} + [\bar{8}\cdot 32]}\, s_{12}\, .$$

The quantity $r_{12} = J_{12}/2\,E_{12}$, the density of the ν_{12} radiation relative to the blackbody radiation, decreases in the atmosphere from an interior value near 1 to the observed low boundary value. Replacing it by the variable y, and putting $F_{12}/2\,E_{12}=z$ we have the equations

$$\frac{dy}{s_{12}\,dx} = \left(1 + \frac{k}{s_{12}}\right) z\, ; \quad \frac{dz}{s_{12}\,dx} = -\frac{k}{s_{12}}\,(1-y) - [\bar{6}\cdot 79]\, \frac{1-y}{y + [\bar{8}\cdot 32]}\, .$$

These equations have been solved by numerical integration (after the principles indicated in my former paper), with the boundary conditions $y = z$ for $x = 0$, and y and z for $x = \infty$ approaching asymptotically to 1 and to a value near 0. The value of s_{12} was assumed to be so large, i.e. the fraction k/s_{12} so small, that without the cyclical term the intensity was only 0·007. It appeared that compared with the cyclical term the influence of the terms with k was almost entirely negligible; the variation of $2\,E$ with depth by the increasing temperature was negligible too. The result for the intensity at the surface was 0·047. It represents the brightening up of the otherwise dark centre of a hydrogen $H\alpha$ line as a consequence of the most important of the cyclical transitions between the different states of the atom.

Some remarks must be added. There are still six other possible cyclical transitions which can be treated in the same way. They are, however, interlaced, because in hydrogen all transitions between two levels of the same total quantum number have materially the same frequency ν, and the r of the resulting line is determined by the sum total of all these cycles. With other atoms the different transitions which in hydrogen are combined, are separated into different lines and so such cycles have to be treated separately. With hydrogen all the seven cycles are contributing to the intensity in the same absorption line ν_{12}, each working on the fraction $A_{21}/\Sigma A_{21}$ of all the transitions ν_{12}; hence their G for the first two are multiplied with 2·26/9·36, for the next two with 0·63/9·36 and for the last three with 6·47/9·36. In the equation for dF_{12}/dx instead of one G term we have now seven $\Sigma G\nu_{12}\, A_{21}/\Sigma A_{21}$;

$$\sum \frac{A_{21}}{\Sigma A_{21}}\, G\nu_{12} = \sum \frac{A_{21}}{\Sigma A_{21}}\, G\, \frac{s_{12}}{\alpha_{12}}\, \sigma_{12} = \sum \frac{A_{21}}{\Sigma A_{21}} \cdot \cdot \frac{\Sigma s_{12}}{\Sigma \alpha_{12}} \cdot G\sigma_{12}\, ,$$

where Σs_{12}, the total monochromatic absorption coefficient, is the quantity s_{12} in the former formulae. With the same suppositions, that for all these cycles G is small, and r_{23}, r_{34} and r_{14} may be taken 1, the coefficients of the equations for their G were computed numerically. The resulting equation is

$$\frac{dz}{s_{12}\,dx} = -\frac{k}{s_{12}}\,(1-y) - (1-y) \left\{ \frac{2\cdot 00}{y + 0\cdot 051} + \frac{0\cdot 0033}{y + 0\cdot 00008} + \frac{0\cdot 205}{y + 0\cdot 646} + \frac{1\cdot 15}{y + 0\cdot 240} + \frac{0\cdot 408}{y + 0\cdot 0023} + \right.$$
$$\left. + \frac{3\cdot 72}{y + 0\cdot 021} + \frac{1\cdot 55}{y + 0\cdot 457} \right\} 10^{-4}\, ,$$

where the terms are arranged in the order of the cycles on p. 155. In order to show their relative importance their values (with the factor $1-y$ included) are given here for

$$y = 0\cdot 050 \quad (188 + 1 + 3 + 38 + 74 + 498 + 29)\; 10^{-5} = 0\cdot 00830$$
$$y = 0\cdot 200 \quad (\;\;64 + 0 + 2 + 21 + 16 + 135 + 19)\; 10^{-5} = 0\cdot 00257$$
$$y = 0\cdot 700 \quad (\;\;12 + 0 + 1 + \;\;5 + \;\;3 + \;\;24 + \;\;6)\; 10^{-5} = 0\cdot 00051$$

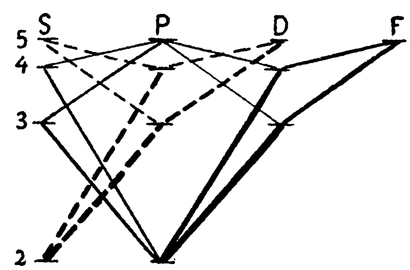
The resulting surface value for y and z by numerical integration is found 0·0497, not much different from what the strongest cycle in the strongest component alone would produce.

This value does not depend on s_{12} itself; it represents the intensity in the centre of the line, on the suppositions adopted, if without cyclical transitions this intensity would be zero or nearly so. This does not mean, however, that in the real case of the solar $H\alpha$ this central value must be present. If the number of active atoms increases steadily and so also s_{12} increases, the lines produced by the other transitions will become stronger, their residual intensities r too fall to values near zero, and the suppositions made above do not hold any more. Then the difference $r_{14} - r_{12}\, r_{23}\, r_{34}$ cannot be represented by $1 - r_{12}$ but decreases to a very small factor, especially by the decrease of r_{14}. It is clear that a transition cycle of this kind can only fulfill its function of brightening the centre of the line ν_{12}, if the radiation J_{12} is already nearly extinguished, while all the others, especially J_{14} keep their full original strength; as soon as J_{14} becomes extinguished too, the cycle ceases. The transition ν_{14} produces the $H\gamma$ line; when $H\gamma$ and $H\alpha$ both are strong lines, as in the solar spectrum, there will be no appreciable brightening of $H\alpha$. Theoretically the broadening of the absorption-coefficient curve by the Stark effect tends to keep a high value of J_{14} for the higher members of the Balmer series; its effect in $H\gamma$, however, is not yet large enough to invalidate the foregoing statement.

It must still be remarked, that while in this cycle $H\alpha$ would be brightened by energy coming from $H\gamma$ absorptions, $H\gamma$ should be darkened by the same cycle. On the other hand $H\gamma$ should be brightened by other cycles running through still higher levels. Thus all these different levels and cycles are interlocked. In the same way the $H\alpha$ transition may be part of a lower cycle 1 S — 2 P — 3 D — 4 P — 1 S; since, however, the intensity of the solar radiation in the Lyman wavelengths 1 S — 2 P and 1 S — 4 P is very small, this cycle will not give any appreciable result.

7. We have now to consider transition cycles of the second kind, where the highest level 4 is reached by two jumps from each side. There are 7 such cycles possible; we write here only the strongest one, which consists entirely of transitions of increasing azimuthal quantum number

2 P 6·47 3 D 0·455 5 F 0·261 4 D 2·07 2 P



The cycle equation here takes the form

$$\left(\frac{n_2 g_1}{n_1 g_2}\sigma_{12}-J_{12}\right)s_{12}/\nu_{12} = \left(\frac{n_4 g_2}{n_2 g_4}\sigma_{24}-J_{24}\right)s_{24}/\nu_{24} = -\left(\frac{n_4 g_3}{n_3 g_4}\sigma_{34}-J_{34}\right)s_{34}/\nu_{34} = -\left(\frac{n_3 g_1}{n_1 g_3}\sigma_{13}-J_{13}\right)s_{13}/\nu_{13} = G,$$

which, after elimination of the numbers, becomes

$$G^2\left(\frac{1}{\alpha_{12}\alpha_{24}} - \frac{1}{\alpha_{34}\alpha_{13}}\right) + G\left(\frac{j_{12}}{\alpha_{24}} + \frac{j_{24}}{\alpha_{12}} + \frac{j_{34}}{\alpha_{13}} + \frac{j_{13}}{\alpha_{34}}\right) + j_{12}j_{24} - j_{34}j_{13} = 0.$$

Introducing the numerical values in the same way as in § 5 we have

$$\alpha_{12} = [1{\cdot}03]; \quad \alpha_{24} = [\bar{8}{\cdot}11]; \quad \alpha_{34} = [\bar{7}{\cdot}43]; \quad \alpha_{13} = [\bar{0}{\cdot}53].$$

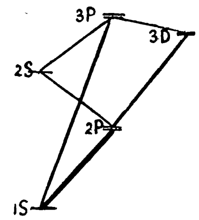
$$-G^2\,[2{\cdot}01] + G\left\{[0{\cdot}20]\,r_{12} + [\bar{8}{\cdot}28]\,r_{24} + [\bar{9}{\cdot}22]\,r_{34} + [0{\cdot}44]\,r_{13}\right\} + [\bar{7}{\cdot}62]\,(r_{12}\,r_{24} - r_{34}\,r_{13}) = 0.$$

Here the case is more difficult than in the first kind of cycles, because we have to do with the difference of two nearly equivalent transitions. Both 2 P — 3 D and 2 P — 4 D are strong transitions, chief components of $H\alpha$ and $H\beta$. So r_{12} and r_{13} will become small at the same time and of not much different order, so that the difference in the last term will be small. We may assume that on this account G will be small and the term with G^2 is negligible; if then, moreover, we assume the other lines to be weak, i.e. r_{24} and $r_{34} = 1$, we have

$$G = \frac{[\bar{7}{\cdot}62]\;(r_{13} - r_{12})}{[0{\cdot}20]\,r_{12} + [0{\cdot}44]\,r_{13} + [\bar{9}{\cdot}27]}$$

As to the numerical coefficients G comes out rather larger than in the former case. It is, however, lowered by the unknown factor $r_{13} - r_{12}$, which certainly is much smaller than $1 - r_{12}$. To find its value a set of simultaneous equations for J_{12} and J_{13} must be solved, which offers some difficulties, not only mathematically, but also materially, because the transition 2 P — 4 D itself is involved in other cycles, where it is the lowest component 1 — 2. It does not seem probable that the resulting intensity by this transition cycle will exceed the result from the other cycles; an exact treatment will be necessary to settle this point.

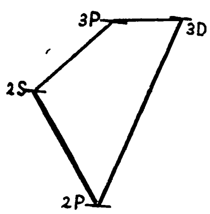
8. Returning now to UNSÖLD's first case of the yellow sodium doublet 1 S — 2 P, we have, as the most important, two cycles of transitions capable to throw light into the dark centre of these lines viz 1 S — 3 P — 2 S — 2 P — 1 S and 1 S — 3 P — 3 D — 2 P — 1 S. The transition probabilities for 1 S — 2 P and 1 S — 3 P have been computed by SUGIURA [1]); his values $\cdot 6228.10^8$ and $\cdot 0297.10^8$ have the ratio 21·0. The other probabilities are not known; qualitatively we may assume that compared with the computed ones 2 P — 2 S will be small and 3 D — 3 P very small. Taking the first with 2 S as the strongest cycle, in which the transition 2 — 3 is the most difficult, then in the equation for G the coefficient of $r_{12}\, r_{34}$ will be the largest, and with the same approximations as used before we have



$$\frac{1}{s_{12}\,.\,2E_{12}}\, G\nu_{12} = \frac{\alpha_{23}}{\alpha_{12}}\, e^{-h(\nu_{23}-\nu_{12})/kT}\, \frac{1-r_{12}}{r_{12}+b} = \frac{A_{23}}{A_{12}}\, \frac{g_3 g_1}{g_2 g_2}\, e^{-h\nu_{23}/kT}\, \frac{1-r_{12}}{r_{12}+b}$$

where b is a small constant, depending on the other coefficients in the equation. If we suppose A_{23}/A_{12} to be below 0·01 (probably it is much smaller) the coefficient is of the order 10^{-4} and judging from the computations for hydrogen the resulting intensity will be of the order 0·01 only.

For the green Mg triplet 2 P — 2 S the matter is more complicated. In the cycle of transitions 2 P — 2 S — 3 P — 3 D — 2 P the transition 2 P — 2 S is not of a larger order of magnitude than 2 P — 3 D, because in the first the azimuthal quantum number decreases; they produce the green triplet 5167, 5172, 5183 and the violet triplet 3829, 3832, 3838, consisting both of strong lines. The relative transition probabilities have been determined experimentally by MINNAERT and MULDERS [2]); for line $P_0 - S_1$ (5167) of the green and line $P_0 - S_1$ (3829) of the violet triplet they find $A_{41}/A_{21} = 0{\cdot}17$. We will treat only these two lines as specimens of the two triplets, and neglect the interaction of the different multiplet members; then in the same suppositions as before the simplified cycle equation is



$$G\left(\frac{j_{12}j_{23}}{\alpha_{34}} + \frac{j_{12}j_{34}}{\alpha_{23}} + \frac{j_{23}j_{34}}{\alpha_{12}} + \frac{1}{\alpha_{14}}\right) = (j_{14} - j_{12}\,j_{23}\,j_{34}).$$

With

$$j_{12} = [-\,2{\cdot}00]\, r_{12}; \quad j_{23} = [-\,0{\cdot}68]\, r_{23}; \quad j_{34} = [-\,0{\cdot}03]\, r_{34}; \quad j_{14} = [-\,2{\cdot}71]\, r_{14},$$

and with $\alpha_{12} = 1, \alpha_{14} = 0{\cdot}17$ we have (for $r_{23} = r_{34} = 1$)

$$\frac{1}{s_{12}.2E_{12}}\, G\nu_{12}\left\{\left(\frac{[\bar{7}{\cdot}32]}{\alpha_{34}} + \frac{[\bar{7}{\cdot}97]}{\alpha_{23}}\right) r_{12} + [0{\cdot}06] + [0{\cdot}00]\right\} = \frac{[2{\cdot}00]}{[\bar{9}{\cdot}23]}\, [\bar{7}{\cdot}29]\ (r_{14} - r_{12}).$$

The smallness of the unknown probabilities α_{23} and α_{34} determines the term with r_{12} that will occur in the denominator of G. If we write it in this form

$$G\nu_{12} = [\bar{9}{\cdot}73]\ \frac{r_{14} - r_{12}}{1 + b\,r_{12}}$$

it appears that as to the numerical coefficient the mutual influence of the two lines is very large; α_{23} or α_{34} must fall very much below 0·001 to give a perceptibly smaller influence. Since, however, r_{14} and r_{12} are of the same order of magnitude, the real influence may be small. Here again we have the case that r_{12} and r_{14} are two unknown quantities that must be found by solving two simultaneous equations of the second order.

Summarizing we may say, that a brightening of the centre of Fraunhofer lines by higher levels, by means of cyclical transitions, is possible; that in some special favourable cases a central intensity of the order 0·05 may be produced; but that in all the real cases considered of strong solar lines the effect is hardly perceptible, say, of the order 0·01.

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1) *Philos. Magazine* **4.** 495 (1927).
2) *Zsch. f. Astrophysik* **2.** 176 (1931).