The Theoretical Contours of Absorption Lines. (Second Paper.)

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In our former paper under this title * we treated the case of an atmosphere of constant composition. Now in reality the different layers of an atmosphere cannot contain the same fraction of atoms of a certain kind, because the ionization changes with pressure and temperature. The most important factor is the change of electron pressure, from zero at the top steadily increasing downward. The change of ionization by the increase of temperature in the deeper layers plays a far less important rôle and will be neglected here.

The method of numerical integration described in the former paper (I., p. 144) can be applied to any case of a particular line in a certain star (with given temperature, gravity, abundance, ionization potential), as soon as the number of active atoms (neutral or ionized) in each layer by means of definite assumptions is computed. For the total contour of such a line this would involve the same labour as in the case, already treated, of an atmosphere of constant composition. If we wish to derive general results, it will be necessary, therefore, to keep the special parameters of each special case outside the process of numerical integration. The atmospheric layers which contribute the main part, say 80 to 90 per cent., to the formation of an absorption line, do not extend over a large range of pressure, not larger usually than between a certain P and 10P. Over such a range, part of the factors may be assumed to be constant, and the variation of the state of ionization can be expressed by a simplified formula.

The ionization formula

$$\frac{x_1}{x_0}\mathrm{P} = 10^{-5039\mathrm{V}_1/\mathrm{T}}\mathrm{T}^{2\cdot5}10^{-0\cdot48},$$

where x_1 and x_0 denote the ionized and the neutral fraction, and the electron pressure P is given in dynes/cm.^{2}, can be written, in Milne's notation,

$$x_1/x_0 = \mathrm{K}/\mathrm{P},$$

where K is a function of the temperature, containing only the ionization potential.

If $\mathrm{K}/\mathrm{P} > 10$, then $x_1/x_0 > 10$, and the fraction ionized is between ·90 and 1·00. Because the main part is ionized, for the neutral fraction we may use the simplified formula

$$x_0 = \mathrm{P}/\mathrm{K}.$$

The concentration of neutral atoms in the deeper layers increases regularly with the electron pressure. At the same time the concentration of the ionized atoms may be taken as constant.

* *M.N.*, **91**, 139, 1930 (which will be quoted as I.).

If, on the contrary, $K/P < 0{\cdot}10$ the fraction x_0 is between 0·9 and 1·0; now the concentration of neutral atoms may be taken the same for all depths, whereas for the ionized atoms we have

$$x_1 = K/P.$$

Here we have the case that the concentration of ionized atoms decreases with increasing depth. When by increasing temperature the atoms begin to ionize they behave in this way. After the first ionization is complete the second ionization begins to lower the number of singly ionized atoms. Since, however, the difference between two succeeding ionization potentials usually is rather large, the ratio of their values of K will amount to several powers of 10, and the range of pressures, over which the fraction of this state of ionization remains above 90 per cent., will be large.

Hence we have to distinguish the following three extreme cases:—

A. The concentration increases $\sim P$. This case is represented by the *Na* doublet in the sun, by the Balmer lines in the B stars, by the Ca^+ lines in the A stars, and more generally, where a line is already rapidly decreasing with increasing temperature.

B. The concentration is constant. This holds for lines of neutral atoms at low temperature (hydrogen in G and K stars, iron lines in K stars) and for resonance lines of ionized atoms at their maximum intensity (Ca^+ doublet in the sun).

C. The concentration decreases $\sim 1/P$. We meet this case with enhanced lines in their first appearance, such as Fe^+ lines in the sun and He^+ lines in the O stars.

Case B has been dealt with in our first paper; the others will be treated here.

2. *Case A.* The formula (8) for the monochromatic absorption coefficient (I., p. 150), expressing it in $\Delta\lambda$, and putting $b' = b\lambda_0^2/2\pi c$, may be written:

$$s_0 = \left\{\frac{2\pi\epsilon^4}{3m^2c^4m_H}\left(\frac{\lambda_0}{\Delta\lambda}\right)^2 + \frac{2\pi\sqrt{\pi}\epsilon^2}{mcbm_H}e^{-(\Delta\lambda/b)^2}\right\}\frac{n}{\mu}.$$

For the fractional abundance n we substitute n_0P/K. We denote the result by $s_0 = \sigma P$, and substitute it into the equations (I., p. 143)

$$\frac{dy}{d\xi} = (k_0 + s_0)z; \qquad \frac{dz}{d\xi} = \left(k_0 + s_0\frac{P}{P+Q}\right)(y - 2E).$$

Putting

$$dx = k_0 d\xi = \kappa_0\frac{p'}{g}P\,dP; \qquad x = \tfrac{1}{2}\kappa_0\frac{p'}{g}P^2; \qquad q' = \tfrac{1}{2}\kappa_0\frac{p'}{g}Q^2,$$

we have for the optical depth which determines the temperature

$$\tau = \tfrac{1}{2}\bar{\kappa}\frac{p'}{g}P^2 = \frac{\bar{\kappa}}{\kappa_0}x,$$

and

$$\frac{dy}{dx} = \left(1 + \frac{\sigma}{\kappa_0}\right)z; \quad \frac{dz}{dx} = \left(1 + \frac{\sqrt{x}}{\sqrt{x} + \sqrt{q'}}\frac{\sigma}{\kappa_0}\right)\left(y - 2E_0\left(1 + c\frac{\bar{\kappa}}{\kappa_0}x\right)\right) \quad (1)$$

In the wings of an absorption line, where the collision part of the monochromatic coefficient may be neglected compared with the continuous absorption, these formulæ reduce to the simpler form

$$\frac{dy}{dx} = \left(1 + \frac{\sigma}{\kappa_0}\right)z; \quad \frac{dz}{dx} = y - 2E_0\left(1 + c\frac{\bar{\kappa}}{\kappa_0}x\right) \quad . \quad (2)$$

The coefficients are constant, because the monochromatic and the continuous absorption both increase proportionately to the electron pressure. The equations can be integrated now in a finite form. Putting $1 + \sigma/\kappa_0 = n^2$ we find

$$y_0 = z_0 = 2E_0\frac{1 + c\frac{\bar{\kappa}}{\kappa_0}/n}{1 + n}.$$

The continuous background has the intensity $E_0\left(1 + c\frac{\bar{\kappa}}{\kappa}\right)$, so that the residual intensity is

$$r = \frac{2}{1 + n}\frac{1 + c\frac{\bar{\kappa}}{\kappa_0}/n}{1 + c\frac{\bar{\kappa}}{\kappa_0}} \quad . \quad . \quad . \quad . \quad (3)$$

For $r = \frac{1}{2}$ this formula gives the value of n corresponding to the half-width of a line. It depends on the value of $c\frac{\bar{\kappa}}{\kappa_0}$ assumed. For

$c\frac{\bar{\kappa}}{\kappa_0} =$	0·8	1·0	1·2	1·5	2·0	3·0
$n =$	2·078	2·0	1·942	1·878	1·808	1·732
$\frac{\sigma}{\kappa_0} =$	3·32	3·0	2·77	2·53	2·27	2·00.

Substituting in the formula for s_0 the values of the physical constants, we find (I., p. 154)

$$s_0 = [-1{\cdot}00]\frac{n}{\mu}\left(\frac{\lambda_0}{\Delta\lambda}\right)^2$$

and

$$\sigma = [-1{\cdot}00]\frac{1}{K}\frac{n_0}{\mu}\left(\frac{\lambda_0}{\Delta\lambda}\right)^2.$$

Taking, as in our former computations, $c\frac{\bar{\kappa}}{\kappa_0} = 1{\cdot}5$, hence $\frac{\sigma}{\kappa_0} = 2{\cdot}53$,

and assuming $\kappa_0 = [-2{\cdot}1]$, the largest possible theoretical value (I., p. 166), the expression for the half-width for $r = \frac{1}{2}$ becomes

$$\left(\frac{\Delta\lambda}{\lambda_0}\right)^2 = [0{\cdot}70]\frac{1}{K}\frac{n_0}{\mu} \quad . \quad . \quad . \quad (4)$$

We apply this formula to the D_1D_2 doublet of *Na* in the solar spectrum. With $T = 6000°$, $V_1 = 5{\cdot}12$ we find $K = [4{\cdot}70]$. For $\Sigma(\Delta\lambda)^2 = 0{\cdot}116$ (I., p. 154) we find $\frac{n_0}{\mu} = [-4{\cdot}48]$, $n_0 = [-3{\cdot}12]$. This is the abundance of all the sodium atoms, neutral or ionized, which, therefore, constitute 0·076 per cent. in mass, $\frac{1}{300}$ per cent. in number of the total atmosphere. In our former treatment, where a constant abundance of neutral atoms throughout the atmosphere was assumed, a total abundance $[-2{\cdot}6]$ was found. The present result is, of course, more reliable.

3. For the darkest centre of a line we may assume just the reverse: the continuous absorption is negligible compared with the collision part of the monochromatic absorption. Then the same method may be used as in I., § 6; the general solution of the differential equation is a definite integral. We now put

$$x = \tfrac{1}{2}\sigma\frac{p'}{g}P^2; \qquad q = \tfrac{1}{2}\sigma\frac{p'}{g}Q^2.$$

(This q is for $\sigma Q = s_0$ identical with q in I.) Then the solution of

$$\frac{d^2y}{dx^2} = \frac{\sqrt{x}}{\sqrt{q}}(y - 2E_0)$$

is given by

$$y - 2E_0 = B\int_1^\infty e^{-tp}(p^2-1)^{-\frac{9}{10}}dp$$

$$z = -B(\tfrac{4}{5})^{-\frac{1}{5}}\int_t^\infty e^{-u}(u^2-t^2)^{-\frac{9}{10}}u\,du,$$

where $t = \frac{4}{5}q^{-\frac{1}{4}}x^{\frac{5}{4}}$ and $pt = u$.

For the boundary $x = 0$, $t = 0$ these integrals are

$$y_0 - 2E_0 = B\int_1^\infty (p^2-1)^{-\frac{9}{10}}dp = BI_1$$

$$z_0 = -B(\tfrac{4}{5}q)^{-\frac{1}{5}}\Gamma(\tfrac{1}{5}) = -Bq^{-\frac{1}{5}}I_2,$$

from which, by the condition $y_0 = z_0$, we find

$$y_0 = z_0 = 2E_0\frac{I_2}{I_2 + I_1\sqrt[5]{q}}.$$

For the integrals in this formula we find

$$I_1 = \int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt[5]{\sin^4\phi\cos\phi}} = 6{\cdot}018 \quad \text{and} \quad I_2 = 4{\cdot}389.$$

hence

$$y_0 = z_0 = \frac{2E_0}{1 + 1{\cdot}37\sqrt[5]{q}} \quad . \quad . \quad . \quad (5)$$

4. *Case C.* In this case we have for the concentration of active atoms:

$$n = n_0 K/P.$$

Indicating by s_0^0 the coefficient of monochromatic absorption in the case of a concentration n_0, we have $s_0 = s_0^0 K/P$. Introducing it into the equations of § 2, where again we put

$$d\xi = \frac{p'}{g}dP\,; \qquad k_0 = \kappa_0 P\,; \qquad \tfrac{1}{2}\kappa_0\frac{p'}{g}P^2 = x\,;$$

we find

$$\frac{dy}{dx} = \left(1 + \frac{s_0^0 K}{\kappa_0 P^2}\right)z\,; \qquad \frac{dz}{dx} = \left(1 + \frac{s_0^0 K}{\kappa_0 P(P+Q)}\right)(y - 2E).$$

Now a difficulty arises for the topmost layers of the atmosphere, in consequence of $P^2 = 0$ appearing in the denominator and producing a logarithmic infinity. It is, however, not essential, and is solely due to the numerical approximation introduced; the quantity $n = n_0 K/P$, which for $P = 0$ would become infinite, is practically restricted to the cases $K << P$. The difficulty can be removed by making use of the exact expression

$$\frac{n}{n_0 - n} = \frac{K}{P}\,; \qquad n = n_0\frac{K}{P+K}.$$

Then the equations become

$$\frac{dy}{dx} = \left(1 + \frac{s_0^0 K}{\kappa_0 P(P+K)}\right)z\,; \qquad \frac{dz}{dx} = \left(1 + \frac{s_0^0 K}{\kappa_0 (P+K)(P+Q)}\right)(y - 2E).$$

Putting now

$$\tfrac{1}{2}\kappa_0\frac{p'}{g}Q^2 = q'\,; \qquad \tfrac{1}{2}\kappa_0\frac{p'}{g}K^2 = k\,; \qquad \frac{1}{2}\frac{p'}{g}K s_0^0 = \sigma'$$

they take the form

$$\frac{dy}{dx} = \left(1 + \frac{\sigma'}{x + \sqrt{kx}}\right)z\,;$$

$$\frac{dz}{dx} = \left[1 + \frac{\sigma'}{(\sqrt{x} + \sqrt{k})(\sqrt{x} + \sqrt{q'})}\right]\left[y - 2E_0\left(1 + c\frac{\bar{\kappa}}{\kappa_0}x\right)\right] \quad . \quad . \quad (6)$$

The numerical difficulty of the integrand in the first interval from $x = 0$ coming down from infinity can be overcome by starting the integration not at $x = 0$ but at some low value of x. This procedure of omitting the lowest interval even corresponds to the physical reality, because in the uppermost layers, at the lowest pressure, the number

of active atoms, after reaching a maximum, must go down by second ionization.

The equations in this form do not hold solely for Case C, but so far are quite general. We want them, however, only for the case that $K < P$, *i.e.* $k < x$. At the same time $x << q'$, because the P in the layers most contributing to the absorption are far smaller than Q. Then the fraction $\frac{\sqrt{x}}{\sqrt{x}+\sqrt{q'}}$ of the monochromatic absorption, which is removed by collisions, is small and may be neglected compared with the continuous absorption, especially in the wings of the line. In this case, where only one parameter has to be assumed in the computations, the formulæ are simplified to

$$\frac{dy}{dx} = \left(1 + \frac{\sigma'}{x+\sqrt{kx}}\right)z\,; \qquad \frac{dz}{dx} = y - 2E_0\left(1 + c\frac{\bar{\kappa}}{\kappa_0}x\right). \qquad (7)$$

It may be remarked that the coefficient k, though necessary at the lower limit of x, has only a small influence upon the result; this means that the radiated intensity y_0 depends only on σ', which does not contain the continuous absorption coefficient. We can understand this by the consideration that in this case chiefly the uppermost layers absorb the continuous light coming chiefly from deeper layers; thus κ_0 appears only in the scale of the integration steps and not in the result.

The numerical integrations have been performed with $k = 10^{-2}$ (corresponding for solar conditions to $K = 10^2$) for three different values of σ'. The results are

$\sigma' = 0{\cdot}1$	$y_0 = z_0 = 2E_0\left(0{\cdot}432 + 0{\cdot}404c\frac{\kappa}{\kappa_0}\right)$;	$r = 0{\cdot}830$
1	0·198 0·129	·313
10	0·0403 0·00522	·038.

The residual intensity was computed with $c\frac{\bar{\kappa}}{\kappa_0} = 1{\cdot}5$. This residual intensity is 0·5 for $\log\sigma' = -0{\cdot}35$; hence for the half-width we have

$$\frac{1}{2}\frac{p'}{g}Ks_0{}^0 = [-0{\cdot}35], \qquad s_0{}^0 = [-1{\cdot}00]\frac{n_0}{\mu}\left(\frac{\lambda_0}{\Delta\lambda}\right)^2,$$

or

$$\left(\frac{\Delta\lambda}{\lambda_0}\right)^2 = [-0{\cdot}95]\frac{p'}{g}K\frac{n_0}{\mu} \qquad . \qquad . \qquad . \qquad (8)$$

5. It may be interesting to compare the contours of absorption lines obtained in the three Cases A, B, C. For this purpose the residual intensity was computed for a number of values of σ, s_0, or σ', increasing in the ratio $\sqrt{10}$ (corresponding to a number of values $\Delta\lambda$ on an arbitrary scale decreasing in the ratio $\sqrt{10}$), chosen in such a way that the residual intensity 0·5 is included in the series.

TABLE I.

Comparison of Contours.

Case A.				Case B.		Case C.	
$\Delta\lambda$.	$\log\frac{\sigma}{\kappa_0}$.	n.	r.	$\log a$.	r.	$\log\sigma'$.	r.
3·16	9·403	1·119	0·883	+1·65	0·926		
1·78	·903	1·342	723	+0·65	762	−0·85	0·768
1·00	0·403	1·879	500	−0·35	500	−0·35	·500
0·56	·903	3·0	300	−1·35	258	+0·15	·244
0·32	1·403	5·128	169	−2·35	110	+0·65	·089
0·18	·903	9·0	093	−3·35	048		
0·10	2·403	15·90	052	−4·35	022		

Thus in Case A the contour lines at medium residual intensity are less steep, in Case B they are steeper, and in Case C still somewhat more so. The cause of this difference is easily understood; the active atoms are in Case A more strongly concentrated in the deepest layers, where the outer parts of the wings are formed, so that in the outer wings the line is darker in Case A than in Case B. In Case C, on the contrary, the atoms are more concentrated in the superficial layers and deepen there the inner parts of the absorption line, whereas its outermost parts, produced in the deep layers, are less blackened here. This difference in shape can be expressed in the form of a difference in half-width for various residual intensities. So we find this half-width

for r =	0·25	A. 0·47	B. 0·55	C. 0·57
	0·50	1·	1·	1·
	0·75	1·92	1·72	1·68

The difference is not pronounced, and it is questionable whether in the present state of the photometry of absorption lines in stellar spectra it can be discerned with certainty.

6. The results obtained for the half-width in each of these cases may be used to derive the change of intensity of a line over the series of spectral classes, *i.e.* for a continuous change of temperature. Then we have first to consider the variation with temperature of the different coefficients in our formulæ. The variation of the continuous absorption coefficient κ_0 is not considerable; according to the formulæ, I., § 13, it changes proportionally to $T^{-\frac{3}{2}}$. Much more important, however, is the variation of p', the ratio of the total pressure to the electron pressure.

At lower temperatures only a small fraction of the atoms are ionized; the number of free electrons is small compared with the total number of atoms. Then the continuous absorption coefficient $k_0 = \kappa_0 P$ is also small, the gas is rather transparent, we can see down to a deep level with high total pressure, and the absorption exerted by the great number of atoms is strong. In our formulæ this is expressed by a large value of $p' = p/P$. Because this ratio always presents itself

in the combination g/p', it appears that low temperature works just as a small gravitation in increasing the intensity of the lines.

To compute the relative number of electrons in a stellar atmosphere it would be necessary to know the elementary composition of the atmosphere, *i.e.* the fractional abundance of each kind of atoms. Though a precise knowledge on this point is lacking at present, we may assume, probably, the following qualitative statements: 1. Hydrogen forms by far the most important part of the atmosphere, perhaps 95 to 98 per cent. in mass, and in the first case helium contributes a considerable part to the rest. 2. The metal atoms constitute only some small percentage of the total mass. Then we may deduce the following consequences of this state of things, also qualitatively expressed. (1) As long as the ionization of hydrogen is insignificant (below 8000°) the number of electrons, all of them produced by the metal atoms, remains small. (2) The ionization of hydrogen, between 8000° and 11,000°, raises the number of electrons so much that it equals the number of atoms, and p' is lowered to 2. (3) For higher temperatures this number is not materially increased by multiple ionization of the metal atoms; perhaps the helium atoms may give some increase.

The relative number of electrons is not the same in each layer; in the atmospheric layers it will decrease with the depth. We will neglect here these differences and introduce its mean value for the mean of the layers which contribute chiefly to the absorption. These layers are characterised by a certain value of the optical depth $\int k_0 dx$, which in I., p. 143, is denoted by αx^2. From the numerical integration we find the value of αx^2 for these layers to be nearly 0·4 (log $= -0{\cdot}4$). The relation

$$s_0 \frac{p'}{g} \mathrm{P} = x, \quad \text{hence} \quad \mathrm{P}^2 = \alpha x^2 \frac{g}{p' \kappa_0} \quad \text{or} \quad p' = \frac{0{\cdot}4 g}{\kappa_0 \mathrm{P}^2}$$

connects the coefficient p' with the electron pressure P in that most contributing layer. If we denote the fraction ionized of each constituent of the atmosphere by x_1 and its abundance by A, then the number of electrons, produced by all the constituents, relative to the number of atoms, is given by

$$\frac{\mathrm{P}}{p - \mathrm{P}} = \frac{1}{p' - 1} = \Sigma \mathrm{A} x_1.$$

The ionization x_1 itself depends on the unknown electron pressure P. Expressing it by means of $\frac{x_1}{1 - x_1} \mathrm{P} = \mathrm{K}$, and eliminating p' by the preceding formula, we find an indirect formula for P:

$$\Sigma \mathrm{A} \frac{\mathrm{K}}{\mathrm{K} + \mathrm{P}} = \frac{\mathrm{P}^2}{0{\cdot}4 g/\kappa_0 - \mathrm{P}^2} \quad . \quad . \quad . \quad (9)$$

For a given temperature the function K can be computed for each element. The term $0{\cdot}4 g/\kappa_0$, with $\kappa_0 = [-2{\cdot}1]$ and with $g = [4{\cdot}4]$

(sun) and $g = [2{\cdot}4]$ (giant star) takes the values [6·1] and [4·1]. For the composition of the atmosphere a simplified model is, tentatively, assumed, consisting of a small number of elements with different ionization potentials, which may be considered as representative. In this model the atmosphere consists, in number of atoms, of 0·005 per cent. with ionization potential $V_1 = 4{\cdot}3$ (for *K*), 0·01 per cent. with $V_1 = 5{\cdot}1$ (for *Na*), 0·01 per cent. with $V_1 = 6{\cdot}1$ (for *Ca, Al*), 0·02 per cent. with $V_1 = 7{\cdot}7$ (for *Fe, Mg, Si*), 100 per cent. with $V_1 = 13{\cdot}54$ (for *H*). This corresponds roughly to a mass fraction 0·2 per cent. for *K*, 0·2 per cent. for *Na*, 0·4 per cent. for *Ca, Al*, 1 per cent. for *Fe, Mg, Si*. Perhaps the abundance of atoms with lowest ionization potential is still overestimated here; an error from this reason may be compensated if there is some diffusion of electrons from the deeper and hotter layers into the atmosphere. With these values of A and V_1, and K computed from V_1 and T, the equation (9) was solved by trial and error; the results for P and p' are contained in Table II. (where also the difference in $\log \kappa_0$ with the constant value $-2{\cdot}1$ computed for $T = 6000°$ is given in the second column).

TABLE II.

Electron Abundance.

T.	$\Delta \log \kappa_0$.	$\log g = 4{\cdot}4$.			$\log g = 2{\cdot}4$.		
		log P.	log $\Sigma A x_1$.	log p'.	log P.	log $\Sigma A x_1$.	log p'.
°							
2,000	+·72	−0·47	−7·04	7·04	−1·14	−6·36	6·36
2,500	+·57	+0·31	−5·49	5·49	−0·38	−4·87	4·87
3,000	+·45	0·77	−4·57	4·57	−0·07	−4·22	4·22
3,500	+·35	1·02	−4·07	4·07	+0·12	−3·84	3·84
4,000	+·27	1·15	−3·81	3·81	0·22	−3·66	3·66
5,000	+·12	1·28	−3·54	3·54	0·35	−3·40	3·40
6,000	00	1·43	−3·25	3·25	0·62	−2·86	2·86
7,000	−·10	1·84	−2·41	2·41	1·16	−1·78	1·79
8,000	−·18	2·29	−1·53	1·54	1·58	−0·89	0·94
9,000	−·26	2·60	−0·84	0·90	1·81	−0·29	0·47
10,000	−·33	2·80	−0·35	0·52	1·88	−0·07	0·34
11,000	−·39	2·87	−0·11	0·36	1·89	−0·01	0·30
12,000	−·45	2·89	−0·03	0·32	1·90	0·00	0·30
13,000	−·50	2·90	0·00	0·30	1·90	0·00	0·30

The influence of two distinct sources of free electrons, the easily ionizable metal atoms and afterwards the more resistant hydrogen atoms, is clearly recognized in these figures. Till 6000° (or 5000°) the decrease of p' is slowing down as if a limit will be reached, but then it becomes large again, till the limit 2 is gradually approached.

It must be remarked that the basis of this computation is the assumption that the continuous absorption depends on collisions with electrons only. If collisions of atoms with atoms play a perceptible part in

this absorption in gases with a small number of free electrons, the value for p' for low temperatures should be lowered.

7. The varying behaviour of a line through the series of spectral classes is, in this treatment, separated into three (or two) distinct parts, each represented by a nearly straight line, one ascending (C), one horizontal (B), one descending (A). Between them there are regions of transition where the assumptions underlying each of these cases begin to break down. They must be considered separately.

In Case B the percentage of active atoms in all the layers is assumed to be the same. This does not mean that it is 100 per cent. everywhere. Also when by partial ionization it is lower, the differences between the different layers are not necessarily so large, that the formulæ of Case B are not good approximations. But in this case n, occurring in the formulæ, is lower than n_0 and must be computed by means of the electron pressure in a mean or most contributing layer. In the case that B represents the maximum of ionized atoms, the formulæ for the increasing and the decreasing side are

$$n = n_0 \frac{K_1}{K_1 + P} \qquad \text{and} \qquad n = n_0 \frac{P}{K_2 + P},$$

whereas for a neutral atom there is only a decreasing branch with

$$n = n_0 \frac{P}{K_1 + P}.$$

In the Cases A and C the formulæ contain n_0. But here also a correction outside their proper regions is necessary. In Case A we have used a fractional factor P/K, where the real value should have been P/(K + P). If for the change of this factor with depth a wrong value was assumed, still the result will be nearly right if for its mean amount the right value was taken. Hence in the result the denominator should be replaced by K + P, where for P the electron pressure of the most contributing layer is taken. This will make a difference only in the transition regions. In the same way, in Case C the factor K/P is used instead of K/(K + P); here the result has to be multiplied by P/(K + P) in the transition regions. We see that in these regions the correction factors for B and for the adjacent (C or A) case are complementary.

8. We first apply our results to the behaviour of the Ca^+ lines H and K. The formulæ to be used for the ascending, the maximum, and the descending branches, with their corrections, are

$$\left.\begin{array}{lll} \text{C.} & \left(\frac{\Delta\lambda}{\lambda_0}\right)^2 = [-\cdot 95]\frac{p'}{g} K_1 \frac{n_0}{\mu}; & \text{Corr. } \frac{P}{K_1 + P} \\ \text{B.} & \left(\frac{\Delta\lambda}{\lambda_0}\right)^2 = [-1\cdot 17]\sqrt{\frac{p'}{g\kappa_0}}\frac{n}{\mu}; & n = n_0 \frac{K_1}{K_1 + P} \quad \text{or} \quad n = n_0 \frac{P}{K_2 + P} \\ \text{A.} & \left(\frac{\Delta\lambda}{\lambda_0}\right)^2 = [-1\cdot 40]\frac{1}{\kappa_0 K_2}\frac{n_0}{\mu}; & \text{Corr. } \frac{K_2}{K_2 + P} \end{array}\right\} \quad (10)$$

In Table III. the necessary data and the results are put down. The temperature functions K_1 and K_2 are computed with $V_1 = 6{\cdot}09$ and $V_2 = 11{\cdot}82$; for κ_0 the value $[-2{\cdot}1]$ for $T = 6000°$ is used; for the other temperatures the correction $\Delta\kappa_0$ from Table II. has been applied. As results, the values of the coefficients of $\frac{n_0}{\mu}$ or $\frac{n}{\mu}$ are given, with the corrections or the factors n/n_0 already applied. The computations are made for the sun ($\log g = 4{\cdot}4$) as well as for a giant star with g one hundred times smaller. In the last columns $\Delta\lambda$ itself is given in an arbitrary unit; in the case of H and K, where the contours of the lines are almost entirely determined by the resonance wings, the total absorption of the line and its effective width will be proportional to these values.

TABLE III.

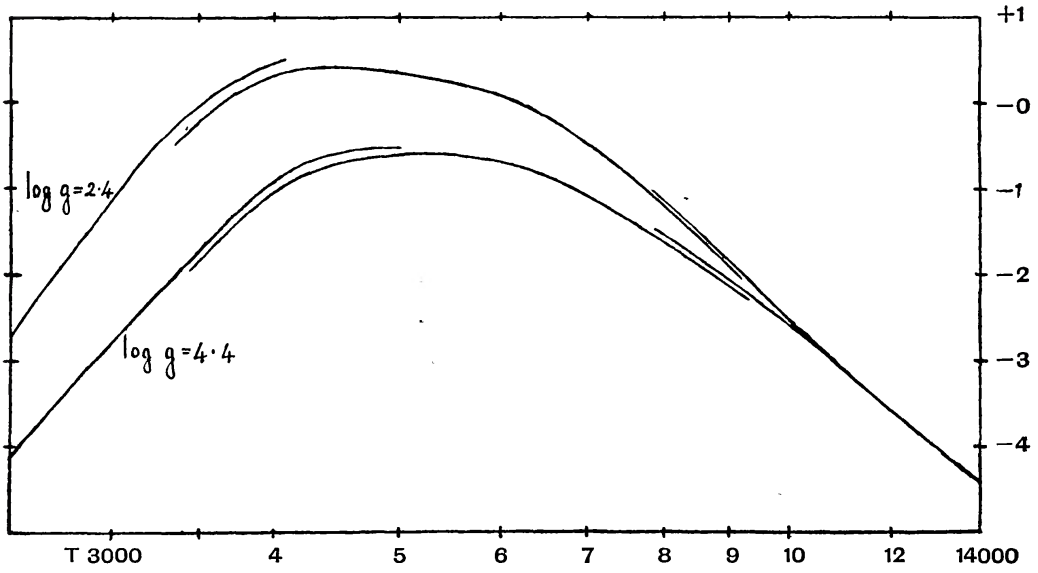
Lines of Ionized Calcium.

T.	K_1.	K_2.	Dwarf. (C).	Dwarf. (A).	Dwarf. (B).	Giant. (C).	Giant. (A).	Giant. (B).	Δλ. Dwarf.	Δλ. Giant.
2,500°	−4·25		−4·11			−2·73			·88	4·3
3,000	−2·02		−2·80			−1·15			4·0	27
3,500	−0·39		−1·69	−1·88		−0·02	−0·20		14·2	98
4,000	+0·86		−0·86	−1·02		+0·44	+0·29		37	140
5,000	+2·63	−3·14	−0·55	−0·63			+0·32		48	145
6,000	+3·82	−0·96		−0·70			+0·10		45	112
7,000	+4·75	+0·62		−1·10			−0·48		28	58
8,000		+1·83	−1·54	−1·59		−1·14	−1·20		16	27
9,000		+2·79	−2·05	−2·15		−1·87	−1·97		9·4	11·6
10,000		+3·56	−2·60			−2·53			5·0	5·4
12,000		+4·76	−3·62			−3·61			1·55	1·55
14,000		+5·63	−4·38			−4·38			·49	·49

The dotted lines separate the regions where the correction is less than $[-0{\cdot}30]$.

The table and the curves show a rather rapid increase to a flat maximum, at 5300° for $g = [4{\cdot}4]$, at 4400° for $g = [2{\cdot}4]$, and a slower decline for higher temperatures. For giant stars the intensity is stronger than for dwarfs, especially at lower temperature; for high-temperature stars the difference disappears. The result for 6000°, viz. $[-0{\cdot}70]\frac{n_0}{\mu}$, corresponds for the sun to a value $\left(\frac{\Delta\lambda}{\lambda_0}\right)^2 = 60/16 \cdot 10^6$ for the H and K lines taken together (I., p. 154); hence we find $\log \frac{n_0}{\mu} = -4{\cdot}73$ and $\log n_0 = -3{\cdot}13$ or $n_0 = 1/1350$. This is much

smaller than was found in my former paper, chiefly because p', which was then assumed to be 2, turns out now to be a thousand times larger. That this large value of p', *i.e.* the extremely small percentage of free electrons and the small continuous absorption of the solar atmosphere, is probably real, may be seen from the curve of the Ca^+ line intensities. With a small value of p' the intensity for the lower temperatures would be strongly diminished and the maximum would be displaced to a temperature of nearly 7000°, certainly not in accordance with observation. The observed low temperature of the maximum of the Ca^+ lines (viz. in the K-type stars) may, in the same way, support an argument that atomic collisions cannot give an important contribution to the continuous absorption when free electrons are scarce.



9. In the spectrum of hydrogen we have to deal with Cases B and A only. For the Balmer lines the number of neutral hydrogen atoms has to be multiplied by the Boltzmann factor. By the increase of this factor the number of atoms in the second state increases with temperature; afterwards it decreases by the ionization of these atoms. Denoting the Boltzmann factor by β,

$$\beta = e^{-V_1/V_T}; \quad \log\beta = -\frac{5039V_1}{T},$$

we have for the half-width of a line

$$\left.\begin{aligned} \text{B.} \quad & \left(\frac{\Delta\lambda}{\lambda}\right)^2 = [-1{\cdot}17]\beta\sqrt{\frac{p'}{g\kappa_0}}\frac{n}{\mu}; \qquad n = n_0\frac{P}{K+P} \\ \text{A.} \quad & \left(\frac{\Delta\lambda}{\lambda}\right)^2 = [-1{\cdot}40]\frac{\beta}{\kappa_0 K}\frac{n}{\mu}; \qquad \text{Corr.}\ \frac{K}{K+P} \end{aligned}\right\} \quad . \quad (11)$$

The data for the computation and the results are found in Table IV. The functions K are computed with $V_1 = 13{\cdot}56$; for gravitation again two values $\log g = 4{\cdot}4$ and $2{\cdot}4$ are used.

TABLE IV.

Balmer Lines of Hydrogen.

T.	log K.	log β.	Dwarf. (B).	Dwarf. (A).	Giant. (B).	Giant. (A).	$\Delta\lambda$. Dwarf.	$\Delta\lambda$. Giant.
4,000°	−8·53	−12·80	−13·35		−12·42		·021	·062
5,000	−4·87	−10·23	−10·84		− 9·91		·48	1·11
6,000	−2·40	− 8·53	− 9·23		− 8·42		2·4	6·2
7,000	−0·62	− 7·31	− 8·38		− 7·70		6·5	14·1
8,000	+0·75	− 6·40	− 7·87		− 7·22	−7·16	11·6	24·5
9,000	+1·83	− 5·68	− 7·49	−7·39	− 6·95	−6·84	18	35·5
10,000	+2·70	− 5·12	− 7·25	−7·14	− 6·99	−6·85	25	37
11,000	+3·42	− 4·65	− 7·26	−7·09		−6·99	28	32
12,000	+4·03	− 4·27		−7·18		−7·15	26	26·5
14,000	+5·02	− 3·65		−7·42		−7·42	19·5	19·5
16,000	+5·77	− 3·20		−7·63		−7·63	15·3	15·3
20,000	+6·86	− 2·56		−7·94		−7·94	10·7	10·7

For the dwarf stars with large gravity the maximum is found to occur at 10,500°, for the giant stars at 9500°. At low temperature the giant stars have stronger absorption lines than the dwarfs; a gravity 100 times smaller works as if the concentration were 9 times more, and the lines, in normal circumstances, would be 3 times as wide. The broadening by the Stark effect, however, spoils these simple relations. If we take account of the increase of the Boltzmann factor by the radiation from the hotter layers at great depth,* which is larger for low than for high temperatures, the theoretical maxima are displaced to a somewhat lower temperature.

Wave-lengths of Oxygen and Nitrogen Lines in the Stellar Region.

By C. S. Beals, D.I.C., Ph.D.

Introduction.—This paper contains the results of an investigation conducted for the purpose of obtaining precise laboratory values of the wave-lengths of oxygen and nitrogen lines which occur in the ordinary stellar region of observation. Originally suggested by Professor A. Fowler, the work was begun at the Imperial College, London, and finished some years later at the Dominion Astrophysical Observatory, Victoria, B.C. A previous investigation by J. S. Clark,† undertaken after Sebastian

* *Cf.* A. Unsöld, *Zeitschr. f. Astrophysik*, **1**, 1, 1930.

† *Ap. J.*, **11**, 332, 1914.